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On exactly marginal deformations of $\mathcal{N} = 4$ SYM and type-IIB supergravity on $AdS_5 \times S^5$

Ofer Aharony*

Department of Particle Physics, The Weizmann Institute of Science
Rehovot 76100, Israel
E-mail: Ofer.Aharony@weizmann.ac.il

Barak Kol

School of Natural Sciences, Institute for Advanced Study
Einstein Drive, Princeton NJ 08540, USA
E-mail: barak@sns.ias.edu

Shimon Yankielowicz

School of Physics and Astronomy
Beverly and Raymond Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv 69978, Israel
E-mail: shimonya@post.tau.ac.il

Abstract: $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group SU(N) ($N \geq 3$) is believed to have two exactly marginal deformations which break the supersymmetry to $\mathcal{N} = 1$. We discuss the construction of the string theory dual to these deformations, in the supergravity approximation, in a perturbation series around the $AdS_5 \times S^5$ solution. We construct explicitly the deformed solution at second order in the deformation. We show that deformations which are marginal but not exactly marginal lead to a non-conformal solution with a logarithmically running coupling constant. Surprisingly, at third order in the deformation we find the same beta functions for the couplings in field theory and in supergravity, suggesting that the leading order beta functions (or anomalous dimensions) do not depend on the gauge coupling (the coefficient is not renormalized).

Keywords: Renormalization Group, AdS-CFT and dS-CFT Correspondence, Classical Theories of Gravity, Supergravity Models.

*Incumbent of the Joseph and Celia Reskin career development chair.

Contents

1. Introduction

A conformal field theory may have exactly marginal deformations which preserve the conformal symmetry. The AdS/CFT correspondence [1, 2, 3, 4] maps such deformations to continuous deformations of AdS solutions of gravitational theories which preserve the conformal isometries.

The AdS/CFT correspondence and its generalizations to non-conformal theories provide an equivalence between gravitational theories and non-gravitational theories. In the AdS case the non-gravitational theories are simply local conformal field theories. Computations which can be performed on both sides of the correspondence have provided many tests of the equivalence between the two sides. The main utility of the correspondence so far has been to learn about strongly coupled non-gravitational theories from gravitational solutions (in particular, from solutions of classical (super)gravity). In this paper we will go in the other direction. We will analyze an issue that is well-understood on the field theory side, and attempt to use it to learn about gravitational solutions and to test the AdS/CFT correspondence (in case there are still unconvinced skeptics).

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One general property of supersymmetric conformal field theories is that many of them have exactly marginal deformations, preserving superconformal invariance (see the paper by Leigh and Strassler [5] and references therein). In order to find a conformal theory one has to solve the constraints for the vanishing of all the beta functions. Since there are as many beta functions as there are couplings, these equations generically will not have a non-zero solution (and if they do, any additional solutions are expected to be isolated). However, in $\mathcal{N} = 1$ supersymmetric field theories in four dimensions (or, more generally, in theories with at least four supercharges) the beta functions are linearly dependent on the gamma functions (the anomalous dimensions). Thus, if for some reason there are fewer gamma functions appearing in the beta function equations than there are couplings, then there will generically exist a manifold of solutions, with a (complex) dimension given by the difference between the number of couplings and the number of equations.

We denote the space of conformal theories on which a specific theory X lies by $\mathcal{M}_c(X)$, to be distinguished from the moduli space of vacua $\mathcal{M}$. Given the central role of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory among gauge theories it is especially interesting to study $\mathcal{M}_c(\mathcal{N} = 4)$. Using the methods described above, it was found [5] that $\mathcal{N} = 4$ SYM with gauge group $\mathrm{SU}(N)$, $N \geq 3$ has, in addition to the complex gauge coupling (which is exactly marginal and preserves the full $\mathcal{N} = 4$ supersymmetry), two extra complex exactly marginal deformations preserving $\mathcal{N} = 1$ supersymmetry. Thus, for $\mathrm{SU}(N)$ gauge groups with $N \geq 3$, $\dim(\mathcal{M}_c(\mathcal{N} = 4 \text{ SYM})) = 2_{\mathbb{C}} + 1_{\mathbb{C}}$. This analysis was carried out at weak coupling (using concepts like the anomalous dimension of the elementary fields), but as the gauge coupling is exactly marginal we do not expect the dimension of $\mathcal{M}_c$ to change as a function of the coupling, so we expect to find the same dimension also at strong coupling.

The AdS/CFT correspondence maps conformal invariance in field theory to $\mathrm{SO}(4, 2)$ isometries in the solution to some gravitational theory, and exactly marginal deformations are mapped to continuous families of solutions with $\mathrm{SO}(4, 2)$ isometries. Our objective is to understand the existence of such families of solutions from the gravity side. In particular, since $\mathcal{N} = 4$ SYM with $\mathrm{SU}(N)$ gauge groups is believed to be equivalent to type-IIB string theory on $AdS_5 \times S^5$, we should have $\mathcal{M}_c(\mathcal{N} = 4 \text{ SYM}) = \mathcal{M}(\text{IIB on } AdS^5 \times S^5)$, and the $AdS_5 \times S^5$ solution should belong to a 3-complex-dimensional family of continuous solutions (one of the complex dimensions corresponds to changing the dilaton and axion, but the other two are not known). In supersymmetric flat compactifications of string theory one often finds such large "moduli spaces" of solutions, parameterized for instance by geometric properties of Calabi-Yau manifolds. However, for AdS compactifications the appearance of such "moduli spaces" is less understood, and it seems that the only continuous deformations which appear in the literature involve either the string coupling constant or the integral of p-form potentials on compact p-cycles.

For large N, weak coupling and strong 't Hooft coupling (large $g_{\mathrm{YM}}^2 N$), supergravity is expected to be a good approximation, and we can study this problem in the supergravity approximation, and look for a continuous family of type-IIB supergravity solutions.[1] At least for small deformations away from the $AdS_5 \times S^5$ solution we expect that supergravity

[1]Not all deformations of conformal field theories map to supergravity modes [6], but the deformations which we will discuss here are mapped to supergravity modes so we can use supergravity to analyze them.

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will still be a good approximation, so that close to the $\mathcal{N}=4$ fixed line we should have $\mathcal{M}_c(\mathcal{N}=4,\, N\to\infty, \lambda_{\rm YM}\gg 1) = \mathcal{M}^{\rm SUGRA}(\text{IIB on } AdS^5\times S^5)$, where $\lambda_{\rm YM}\equiv g_{\rm YM}^2 N$. We will thus construct a family of type-IIB supergravity backgrounds, which are expected to also be good type-IIB string theory backgrounds; it would be interesting to construct the relevant string theories directly, and perhaps to understand our results from the worldsheet point of view.

Ideally, we would like to directly translate the field theory arguments for the existence of exactly marginal deformations to gravity, and to be able to prove their existence also on the gravity side. However, it is not clear how to perform this translation, for instance because we do not know how to identify the anomalous dimensions of the elementary fields on the gravity side (this is problematic because these are not gauge-invariant objects, and they depend on how the fields are normalized). Thus, it is interesting to understand the mechanism that guarantees conformal deformations in gravity, and its relation to the field theory mechanism. We were not able to do this, but we construct an explicit solution for the deformation up to second order in the perturbation, and we discuss the obstruction to conformal invariance on the gravity side. We show that this obstruction has the same form as the obstruction in field theory, coming from the anomalous dimensions, but we have not yet been able to make this relation precise. We can also compute the deviation from conformal invariance when we deform by couplings that are not exactly marginal, namely the beta function. We find that at leading order in the superpotential couplings, this deviation is the same at strong coupling (using supergravity (SUGRA)) and at weak coupling (using perturbation theory). We hope that our results will be useful for a general understanding of exactly marginal deformations in gravitational backgrounds. We are confident that the methods we describe here will allow the analysis of geometries other than $AdS_5\times S^5$, in particular $AdS_4\times S^7$ where the number of the deformations is not known and moreover the field theory side is poorly understood.

Seen in a wider context this is a study of a certain space of conformal theories. Exact descriptions of moduli spaces of *vacua* $\mathcal{M}(X)$, have become numerous in recent years with the improved non-perturbative understanding of supersymmetric theories. Given a conformal theory X, determining $\mathcal{M}_c(X)$ is similar to a "moduli" problem, and it is a natural and fundamental question. Actually this is more than an analogy — the AdS/CFT correspondence maps the space of conformal deformations to the moduli space of gravitational solutions on AdS, and also in the case of perturbative string theory, the space of conformal worldsheet theories maps to the moduli space of the space-time theory. Here we make a first step in studying $\mathcal{M}_c$, and we believe that with our current understanding of exact moduli spaces the tools are available to determine at least some $\mathcal{M}_c$'s completely.

Our problem could have been studied now for some years since the discovery of the AdS-CFT correspondence,[2] and indeed it was studied to low orders in the deformation in [7, 8]. However, it has resisted solution so far.[3] It has turned out to be easier to study

[2]In fact, except for the comparison with field theory, all our computations could have been done already in the 1980's, though it would have been hard to motivate them.

[3]A solution to this problem was presented in [9], but it seems to be singular and is not related to the solutions we construct.

supersymmetric mass deformations of $\mathcal{N}=4$ SYM (starting from [10, 11]), partly because these can be analyzed using the truncation to 5d supergravity (which does not include the deformations we are interested in here). Graña and Polchinski [12] studied the first order deformation in the $AdS_5 \times S^5$ solution for an arbitrary superpotential, a result which we use and extend to higher orders.

Method and summary of results. We start by analyzing the field theory in section 2. Using the form of the 1-loop gamma function at weak coupling and the analysis of [5] we write down an equation for the exactly marginal deformations which is invariant under the SU(3) global symmetry group. This equation is quadratic in the couplings, and we expect to find superconformal solutions if and only if it is satisfied.

We turn to supergravity in section 3. We take our ansatz to be the most general ansatz with SO(4, 2) isometry, and our equations are the supersymmetry (SUSY) variation equations rather than the equations of motion of type IIB since we insist on a supersymmetric background. Actually we require only invariance under four dimensional supersymmetries rather than all superconformal charges, but together with the SO(4, 2) isometries of the ansatz these imply the full SU(2, 2|1) superconformal invariance. One wonders at first whether the $\mathcal{N}=4$ 5d gauged supergravity,[4] which is often used to analyze RG flows, could be useful here, but it turns out that it is not. The gauged supergravity is believed to be a consistent truncation of the 10d IIB supergravity in the $AdS_5 \times S^5$ background, and it is known how to identify its fields at first order with the 10d modes (but not at higher orders). Once we know our deformation at first order, which turns out to be a 2-form potential mode on $\mathbf{S}^5$ in the $(\mathbf{1}, \mathbf{45})$ representation of $SO(4,2) \times SO(6)_R$, we can check whether this mode is retained by the gauged supergravity. However, it is easy to see that it is not, since the SO(6) gauged supergravity does not have scalars in the $\mathbf{45}$ (and the more exotic SO(4, 2) gauging does not have a two-form in the $\mathbf{1}$ either).

Next, we construct the perturbation problem by expanding the fields (and the covariantly constant spinor) in a perturbation parameter h, substituting into the equations and attempting to solve them order by order. Two crucial ingredients are the full form of the SUSY variation equations for type IIB found by Schwarz [13], and the list of AdS_5 scalars in this background (these are our variables) found by Kim, Romans, and van Nieuwenhuizen [14]. The SO(4, 2) isometry restricts our solution to depend only on the $\mathbf{S}^5$ coordinates, and supersymmetry mandates an additional $U(1)_R$, which effectively restricts the problem further to $\mathbf{CP}^2$ through the Hopf fibration.

The first order analysis was performed in [12], and the solutions to the linearized equations $L\,\phi^{(1)}=0$ are precisely the marginal deformations which preserve supersymmetry (ϕ stands for all the fields, and the superscript (1) denotes the order in the perturbation). At higher orders the equation is $L\,\phi^{(k)}=\ldots$, where the right-hand side depends only on quantities of a lower order than k (possibly we have also constraint equations $0=\ldots$). These equations can be solved at each order, except for the zero modes of L and the

[4] This denotes 32 supercharges, which in the supergravity literature is denoted as $\mathcal{N}=8$, even though the minimal 5d supergravity has 8 supercharges.

constraint modes, and the crucial point is whether the right-hand side vanishes for these modes. If these modes on the right-hand side are non-zero at order k then the equations cannot be solved beyond this order, and we say that we have an *obstruction*.

At first it seems like an obstruction can appear at any order so that an exact solution would have to survive all orders, which would be infinitely improbable. However, the correct picture is that any given obstruction can eliminate solutions only once — *"the obstruction only shoots once"*. To understand this we should go back and think about the whole system of non-linear equations, rather than consider the perturbation series. The obstruction amounts to adding a finite number of constraints beyond the linear approximation. We know that generically $\dim(\mathcal{M}_c) = \#(\text{zero modes}) - \#(\text{additional constraints})$. The solution of a higher order constraint (without any linear terms) generically leaves a singularity at the origin. Therefore, once we have an obstruction the original perturbative series does not make sense anymore, and cannot be used to obtain obstructions at higher orders.[5]

In section 4 we turn to an explicit order by order study of the equations. The equations, whose variables are the bosonic supergravity fields and the covariantly constant spinor, happen to have an important $\mathbb{Z}_2$ symmetry. A given field can appear in the perturbation expansion either at odd orders or at even orders, and for the covariantly constant spinor this decomposition coincides with its four dimensional chirality decomposition (namely, the positive chirality spinor gets contributions at even orders, and the negative chirality spinor at odd orders). At first order the 3-form and the negative chirality spinor get turned on. At second order we solve for the complex string coupling,[6] the warp factor, the metric perturbation and the positive chirality spinor. The construction of a second order conformal solution for the cases which are supposed to correspond to exactly marginal deformations may be viewed as a verification of the AdS/CFT correspondence.

At second order we find that a certain potential function which contributes to the metric receives a $\log(r/r_0)$ contribution (where r is the radial coordinate of AdS_5) precisely when the (quadratic) field theory equation for exact marginality is not satisfied. However, this $\log(r)$ dependence seems to disappear from all observables.[7] So, it seems that if we do not impose additional constraints we find a second order solution even when the equation is not satisfied. We explain why this result is consistent with our field theory expectations, and we show explicitly that at third order in the deformation a solution is possible only if

[5]As a simple example to have in mind consider the equation $x^2 - y^2 - x^3 = 0$. At the linear level there are two zero modes x and y. At second order we find an obstruction given by the singularity equation $x^2 - y^2 = 0$, whose solution is $y = \pm x$. Naively, at third order we get an additional obstruction, but in fact the way to continue is to re-organize the perturbation expansion by choosing one of the branches and to expand around it. Then, there are no additional obstructions. For instance, we can choose $y = x + \epsilon(x)$ and solve perturbatively for ϵ, finding $\epsilon(x) = -x^2/2 + \cdots$.

[6]This was done already in [12].

[7]In our computation we do not gauge fix most of the diffeomorphism symmetries of the problem, including diffeomorphisms which do not vanish on the boundary of AdS space. It is possible that in a more careful computation such diffeomorphisms should be constrained, and then we would find a different solution, related by a diffeomorphism to ours, which would explicitly include $\log(r)$ factors breaking conformal invariance at second order.

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the field theory condition for exact marginality is satisfied. This arises from a factorization of our third order result that we do not know how to explain directly from the supergravity point of view.

In section 5 we discuss the third order solution in the case where the exact marginality condition is not satisfied. This solution involves a logarithmically running coupling, and we compare it with the expected result from perturbation theory. We find a surprising agreement between the two, even though the field theory result is derived at weak coupling and the supergravity result is at strong coupling. Our results clearly show that the leading deformation-dependent coefficient in the beta function is independent of the 't Hooft coupling in both limits, and an explicit computation shows that even the numerical coefficient is the same. This suggests that there is some non-renormalization theorem for the leading-order term in the beta function (and in the anomalous dimension), but we have not been able to show this directly (or to relate it to any of the known non-renormalization theorems).

Finally, in appendix A we study the singularity at the origin of $\mathcal{M}_c(\mathcal{N}=4)$, some relevant finite subgroups of SU(3), and SU(3)-invariant coordinates for $\mathcal{M}_c$.

To summarize our results, we find an explicit solution for the deformations at second order and display the appearance of the obstruction. Our analysis, however, does not prove that no other obstructions exist, and that the solution we find can be extended to all orders. It would be interesting to investigate this problem further, and to attempt to find an exact (all-orders) solution for the case of exactly marginal deformations.

Related topics. If $\mathbf{S}^5$ had Einstein deformations (or Sasaki-Einstein deformations if we include SUSY), then these would satisfy the equations of motion of type IIB after taking the direct product with AdS_5. However, it is known that $\mathbf{S}^5$ is Einstein rigid — it has no such deformations (actually non-rigid positive Einstein manifolds are not easy to find — none were known until the 1980's [15], but by now many examples are known [16]). String theory gets around this obstacle by generalizing the Einstein condition, adding to the metric additional fields, and incorporating them in the SUSY variation equations.

The geometry of supersymmetric *solutions* of supergravity with non-zero field strengths is not well understood yet. One would like to generalize the familiar notions from the purely geometric case of covariantly constant spinor, reduced holonomy, complex structure, Kähler geometry and so on. In some special cases this is understood. It is known how to add a non-trivial vector bundle which needs to satisfy the Donaldson-Uhlenbeck-Yau equation. Strominger [17] studied backgrounds with non-zero NS 3-form field strength H_{NS} and found that they have a complex structure, but the associated Kähler form k is not closed, but rather $(\partial-\bar{\partial})k \propto H_{\mathrm{NS}}$. More recently, the conditions for possibly warped "compactifications" (over possibly non-compact manifolds) with non-vanishing field strength for higher rank forms were found in several cases. Note that topological constraints (on the cohomology of the field strength) are not relevant in our case since we are smoothly deforming a topologically trivial background with no 3-form flux (and in any case we have no non-trivial 3-cycles).

Another interesting feature of this problem is the change in the amount of supersymmetry on the moduli space. For weakly coupled string backgrounds in flat space with zero RR field strengths this is forbidden, as can be understood either from worldsheet arguments [18] or from a spacetime approach [19] which assumes only the vanishing of the cosmological constant. Our case clearly circumvents these assumptions. It would be interesting to understand what are the conditions for this to happen.

Open questions. The most immediate application of our work, one which we hope to pursue, is the generalization to $AdS_4 \times S^7$. It would also be interesting to generalize our results to orbifolds of $AdS_5 \times S^5$ [20], which have many more exactly marginal deformations than $AdS_5 \times S^5$, as discussed in [21].

In this paper we study the local structure of $\mathcal{M}_c(\mathcal{N}=4)$ around the $\mathcal{N}=4$ theory — its dimension, the first terms in the expansion, and the proper gauge invariant coordinates. It would be interesting to study also the geometry away from this point, together with its global structure. Some particular points on $\mathcal{M}_c(\mathcal{N}=4)$ were studied in [22, 23, 24] and were argued to be dual to IIB string theory on $\mathbb{Z}_m \times \mathbb{Z}_m$ orbifolds of $AdS_5 \times S^5$ with discrete torsion. These orbifolds have a different topology from $AdS_5 \times S^5$, so supergravity must break down as we go towards these points. One can show that indeed the corrections to supergravity become large before the solutions of [22, 23, 24] start having a good geometrical approximation (with $S^5/(\mathbb{Z}_m \times \mathbb{Z}_m)$ much larger than the string scale). Thus, our results are consistent with all these backgrounds sitting together in one large moduli space, which has different geometrical approximations in different regions.

What is the geometry of $\mathcal{M}_c$ in general? Generally, the scalars in vector and tensor multiplets of 5d $\mathcal{N}=1$ supergravity are known to be valued in a "very special geometry" manifold, while scalars in hypermultiplets are valued in a quaternionic space. Together with the supergravity potential these should determine the moduli space and its geometry (see [25] for the state-of-the-art), at least for small deformations when supergravity is valid. Unfortunately, it does not seem to be possible to truncate the computations to a 5d supergravity theory with a finite number of fields, so this analysis may be rather difficult. Alternatively, it should be possible to derive the geometry of $\mathcal{M}_c$ from field theory. The S-duality group $\mathrm{SL}(2,\mathbb{Z})$ is expected to identify different points on $\mathcal{M}_c$. Other interesting global issues are whether $\mathcal{M}_c$ has singularities other than the $\mathcal{N}=4$ theory, and whether it has any non-trivial topology.

Similar methods may be used to study the *non-supersymmetric* deformation problem as well. Clearly $\mathcal{M}_{c,\text{non-susy}} \supseteq \mathcal{M}_c$, and since we do not know of any symmetry to protect additional exactly marginal directions which would break supersymmetry it is natural to expect an equality to hold. At weak coupling this can be shown from perturbative computations. The way to analyze this at strong coupling would be to expand the equations of motion of type-IIB supergravity (rather than the supersymmetry equations) perturbatively in the deformation. At first order there are many more marginal deformations compared to the supersymmetric case. In order to rule these out as exactly marginal deformations one has to solve for the perturbation series going up to an order where they fail. If one could show at some order (hopefully low, though we show that it has to be at least third order) that all but the superconformal ones are not exactly marginal one would be done. However, if some marginal directions are not eliminated one would need to go to all orders to prove that they are indeed exactly marginal, and it is hard to imagine doing that without supersymmetry.

Another issue is the translation into field theory of our perturbative calculation. In our computation we find that the supergravity fields are corrected in perturbation theory in the deformation, with apparently generic corrections (constrained by conformal invariance and the global symmetries). These corrections involve scalar fields coming from spherical harmonics on the $\mathbf{S}^5$, which appear without the usual radial dependence they would have, had they appeared at first order. It is not clear how to interpret the computed higher order corrections in the field theory.

2. Field theory analysis

The lagrangian for an SU(N) gauge theory with $\mathcal{N} = 4$ supersymmetry is uniquely determined up to the choice of the gauge coupling $g_{\rm YM}$ and the theta angle θ, which can be joined into a complex coupling $\tau \equiv \theta/2\pi + 4\pi i/g_{\rm YM}^2$. For any value of τ the theory is exactly conformal, so this coupling is an exactly marginal deformation. The $\mathcal{N} = 4$ vector multiplet includes a vector field, four adjoint Weyl fermions (in the **4** representation of the SU(4) R-symmetry) and six real scalars φ (in the **6** representation). We can write the lagrangian for $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace notation. The $\mathcal{N} = 4$ vector multiplet splits into an $\mathcal{N} = 1$ vector superfield and three chiral superfields Φ^1, Φ^2, Φ^3 in the adjoint representation. In addition to the usual $\mathcal{N} = 1$ kinetic terms, the $\mathcal{N} = 4$ theory has a superpotential of the form

$$W = \frac{1}{6}\tilde{h}\epsilon_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k)\,, \tag{2.1}$$

with $\tilde{h} = g_{\rm YM}$ (if we choose a canonical normalization for the kinetic terms of the Φ^i). When we write the theory in $\mathcal{N} = 1$ language only an SU(3) $\times$ U(1)$_R$ subgroup of the full SU(4) R-symmetry is manifest, with the SU(3) rotating the chiral superfields Φ^i. Our conventions for SU(3) and SU(4) representations, and the relations between them, are summarized in appendix B.

We are interested in studying additional exactly marginal deformations of this theory. For $N \geq 3$, the (classically) marginal deformations which preserve SUSY include, in addition to the gauge coupling constant τ and the superpotential coefficient $\tilde{h}$ discussed above, ten coefficients appearing in a superpotential of the form

$$W = \frac{1}{3}h_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k) \tag{2.2}$$

with symmetric coefficients h_{ijk}. To check for exactly marginal deformations we can start by analyzing which deformations are marginal at 1-loop. The matrix γ of anomalous dimensions of the fields Φ^i, arising from the wave-function renormalization of these fields, is in the $\mathbf{8} + \mathbf{1}$ representation of SU(3). The non-renormalization theorem for the superpotential implies that (if we rescale the fields so that their kinetic terms remain canonical) the coupling constant running is given by

$$\beta_{h_{ijk}} = -\frac{1}{2}\left(h_{ijl}\gamma^l_k + h_{ilk}\gamma^l_j + h_{ljk}\gamma^l_i\right). \tag{2.3}$$

This is required to vanish for exactly marginal deformations, so the gamma function also has to vanish in this case. At 1-loop order the traceless part of the gamma function, in the **8** representation of SU(3), depends only on the couplings h_{ijk}, so its vanishing provides 8 constraints on these couplings, of the form (assuming that the fields are canonically normalized)

$$\gamma^j_i = -\frac{N^2-4}{64N\pi^2} h_{ikl}\,\bar{h}^{jkl} = 0\,, \tag{2.4}$$

where $\bar{h}^{ijk}$ is the complex conjugate of h_{ijk}, and the equation is to be taken in the **8** representation of SU(3) (the singlet γ^i_i in γ^i_j includes also contributions from the gauge coupling and from $\tilde{h}$). The vanishing of these components of the gamma function gives 8 real constraints (since $\gamma^j_i = (\gamma^i_j)^*$), and by an SU(3) rotation we can remove 8 additional real degrees of freedom, and be left with a 2 complex dimensional space of solutions. For an appropriate choice of $\tilde{h}$ (which can be chosen to be real by a global symmetry transformation) all the beta functions vanish on this space of solutions (at one-loop). By an appropriate SU(3) rotation one can write the general solution to (2.4) in terms of two (complex) coefficients h_1 and h_2, appearing in the superpotential as

$$W = \frac{h_1}{2}\,\mathrm{tr}\left(\Phi^1\Phi^2\Phi^3 + \Phi^1\Phi^3\Phi^2\right) + \frac{h_2}{3}\,\mathrm{tr}\left((\Phi^1)^3 + (\Phi^2)^3 + (\Phi^3)^3\right). \tag{2.5}$$

When (2.4) is not satisfied, we have an anomalous dimension for the chiral superfields at second order in h. However, this anomalous dimension is not a physically measurable quantity, since it can be swallowed into a normalization of the fields; in fact, equation (2.3) for the running coupling constant is usually derived by absorbing the wave function renormalization into the normalization of the fields, after which the coupling constant becomes scale-dependent. This scale-dependence is a physically measurable quantity, and we see that if (2.4) is non-zero it occurs at third order in the deformation parameter h.

In fact, the two deformations described in (2.5) are actually exactly marginal [5] (and not just marginal at 1-loop order as we showed above), so that the SU(N) $\mathcal{N}=4$ SYM theory has two (complex) exactly marginal deformations (in addition to the $\mathcal{N}=4$ flat direction corresponding to changing the coupling). This is because if we look at the $\mathcal{N}=4$ theory deformed by the superpotential (2.5), then supersymmetry forces the (Wilsonian) beta functions of h_1 and h_2 (2.3), as well as those of the gauge coupling g_{YM} and the $\mathcal{N}=4$ superpotential coupling $\tilde{h}$, to be proportional to the gamma function of the fields Φ^i. The particular deformations we discuss preserve a $G_1 \supset \mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry given by the transformations $\Phi^1 \to \Phi^2$, $\Phi^2 \to \Phi^3$, $\Phi^3 \to \Phi^1$ and $\Phi^1 \to \Phi^1$, $\Phi^2 \to \omega\Phi^2$, $\Phi^3 \to \omega^2\Phi^3$, where ω is a cubic root of unity.[8] This symmetry forces the gamma functions of the fields

[8] An additional $\mathbb{Z}_3$ symmetry which multiplies all the Φ^i by ω is also preserved by the deformation. The precise symmetry group G_1 is actually not the direct product of the three $\mathbb{Z}_3$ factors, since their generators do not commute; rather it is an order 27 group generated by generators U, V, ω with the

Φ^i to all be the same and it does not allow them to mix, namely $\gamma^i_j = \gamma\delta^i_j$. Thus, we have a single constraint $\gamma = 0$ which is sufficient to ensure conformal invariance. Since this is a single equation in four variables, we expect generically to have a 3-complex dimensional surface where

$$\gamma(\tau, \tilde{h}, h_1, h_2) = 0\,, \tag{2.6}$$

which corresponds to exactly marginal deformations, namely it is a surface of fixed points of the renormalization group flow.[9]

In general we do not know what the surface of solutions to (2.6) looks like, except that it includes the $\mathcal{N} = 4$ theory $g_{\rm YM} = \tilde{h}$, $h_1 = h_2 = 0$. Our 1-loop analysis showed that at weak coupling we can turn on h_1 and h_2 (and then determine $\tilde{h}$ as a function of $g_{\rm YM}$, h_1 and h_2 from (2.6)). In fact, we can also show that for any value of $g_{\rm YM}$ and to leading order in the deformation away from the $\mathcal{N} = 4$ fixed line, the additional exactly marginal deformations are exactly given by h_1 and h_2. This is because the coupling corresponding to changing $\tilde{h}$ to be different from $g_{\rm YM}$ is, in fact, not marginal on the fixed line for non-zero coupling. The coupling corresponding to changing $g_{\rm YM}$ and $\tilde{h}$ together is an $\mathrm{SU}(4)_R$-singlet, which is in a chiral primary multiplet (it is identified with the dilaton in supergravity), so its dimension is protected to be exactly 4 on the $\mathcal{N} = 4$ fixed line. The operators coupling to h_1 and h_2 are also part of chiral multiplets, whose dimensions do not get renormalized in the $\mathcal{N} = 4$ theory. However, by examining the deformation corresponding to changing $\tilde{h}$ to be different from $g_{\rm YM}$, one can see that it is in fact a component of a non-chiral operator in the **15** of SU(4). It has a non-zero anomalous dimension which can be computed in perturbation theory, and the AdS/CFT duality [1] suggests that for large $g^2_{\rm YM}N$ its dimension is at least of order $(g^2_{\rm YM}N)^{1/4}$, since no field which can be identified with this operator appears in the supergravity spectrum. In the $\mathcal{N} = 1$ language this can be seen from the fact that the field $\epsilon_{ijk}\,\mathrm{tr}(\Phi^i\Phi^j\Phi^k)$ is a descendant, since for non-zero coupling $\{\bar{Q}, \bar{\lambda}^k\} \propto [\Phi^i, \Phi^j]$ (where $\bar{\lambda}^k$ are the fermionic components of the superfields Φ^k). An explicit computation of the beta function of $\hat{h} \equiv \tilde{h}/g_{\rm YM} - 1$ shows that $\beta_{\hat{h}}$ must vanish on the fixed line but $\beta_{\hat{h}}/\hat{h}$, which is the anomalous dimension of the operator coupling to $\hat{h}$ on the fixed line, is generally different from zero.

We conclude that field theory arguments imply that there are two exactly marginal deformations away from the $\mathcal{N} = 4$ fixed line, which correspond (at leading order away from the fixed line) to the $\mathrm{U}(1)_R \times G_1$-preserving couplings h_1 and h_2. Note that the fact that the couplings h_1 and h_2 (if we choose $\tilde{h}$ appropriately) are exactly marginal implies that at all orders in perturbation theory (and even non-perturbatively) the gamma function matrix vanishes if we only turn on these couplings. This in turn means that the gamma function matrix vanishes (for an appropriate choice of $\tilde{h}$) whenever the 1-

relations $U^3 = V^3 = \omega^3 = 1$ and $UV = \omega VU$. We provide an additional discussion of this group in appendix A. There is also an unbroken $\mathrm{U}(1)_R$ symmetry, under which the scalars in the chiral superfields have charge 2/3. In the conformal case this $\mathrm{U}(1)_R$ becomes part of the superconformal algebra.

[9] We are being somewhat imprecise here since it is not clear exactly how to define the gamma function in a gauge-invariant way beyond perturbation theory, but we do not expect this subtlety to affect our result. Our results below can be viewed as evidence that considerations of this type do extend beyond perturbation theory.

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loop term (2.4) (corresponding to the projection of the product $h\bar{h}$ of the $\mathbf{10} \times \overline{\mathbf{10}}$ representations of SU(3) onto the $\mathbf{8}$ representation) vanishes. We will denote this term by $(h\bar{h})_{\mathbf{8}}$.

Our 1-loop analysis shows that there cannot be any additional exactly marginal deformations which preserve supersymmetry.[10] In perturbation theory it is simple to compute the beta functions of the various couplings perturbatively, and to find that the couplings in h_{ijk} which are not exactly marginal are, in fact, marginally irrelevant, due to their non-zero beta function at order h^3. These coupling constants are not asymptotically free, so the corresponding theories do not make sense as a full description of the physics, but one can still use them as an effective description in some range of energies where the couplings are small. In such an energy range the couplings will flow according to their perturbative beta functions, which are proportional to the anomalous dimensions matrix γ_i^j. We will construct below the dual of such a flow, which will enable us to compare in section 5 the anomalous dimensions at leading order in h at weak and strong gauge couplings.

We end our field theory discussion by noting that it is possible to translate the gamma functions we discussed here, which are not gauge invariant, to a gauge-invariant language. One way to do this is just to compute the anomalous dimensions of composite chiral operators like $\mathrm{tr}(\Phi^{i_1}\cdots\Phi^{i_k})$ (in a symmetric SU(3) representation); since there are no short-distance singularities when chiral operators are brought together, the anomalous dimension of these gauge-invariant operators is simply the sum of the anomalous dimensions of their components (with an obvious generalization in case the gamma matrix is not diagonal).

Another way to phrase our results in gauge-invariant terms is to use the relation between the gamma functions which appear in the beta function equations and the global symmetries which are broken by the various couplings. In the absence of a superpotential, a classical gauge theory has a U(1) global symmetry rotating the phase of every chiral superfield in the theory (which is enhanced to U(k) if we have k chiral fields in the same complex representation of the gauge group). Every superpotential coupling breaks some combination of these U(1) symmetries, and the gauge coupling also breaks a combination of U(1) symmetries through the axial anomaly. A superpotential coupling $W = \lambda \prod_i (\Phi_i)^{p_i}$ breaks the linear combination of U(1)'s given by $\Phi_i \to e^{i\alpha p_i}\Phi_i$ (while preserving all orthogonal combinations), while the gauge coupling breaks (through the axial anomaly) the linear combination of U(1)'s given by $\Phi_i \to e^{i\alpha C(r_i)}\Phi_i$, where $C(r_i)$ is the quadratic Casimir of the representation r_i of the gauge group which the field Φ_i is in.

In both cases (assuming that the 1-loop gauge coupling beta function vanishes), if the coupling breaks the linear combination given by $\Phi_i \to e^{i\alpha C_i}\Phi_i$, its beta function is proportional to $\sum_i C_i\gamma_i$ (with obvious generalizations to the case of non-abelian global symmetries). This follows from the superpotential non-renormalization theorem or from the NSVZ formula. If the symmetries broken by some couplings are not independent, their beta functions will be linearly dependent, and this is the essence of the arguments of [5] for exact marginality. If the gamma function γ_i is non-zero, it means that under RG flow the

[10] At least at weak coupling; it is hard to imagine a phase transition changing the dimension of the space of conformal theories as one moves from weak coupling to strong coupling, but perhaps this is not completely impossible.

field Φ_i changes as $\Phi_i \to Z_i\Phi_i$ with $\gamma_i = \partial \log(Z_i)/\partial \log(\mu)$ (for a renormalization scale μ). This rescaling of Φ_i is simply a complexified version of the global U(1) symmetry acting on Φ_i. Thus, having non-zero gamma functions in an $\mathcal{N} = 1$ gauge theory is the same as having a complexified global symmetry transformation acting on the chiral superfields in the theory. The physically meaningful gamma functions are those appearing in the beta functions, and these involve complexified global symmetry transformations by the global symmetry generators which are broken by the couplings in the theory, with coefficients whose dependence on the renormalization scale is given by the gamma function.

In the $\mathcal{N} = 4$ case described above, the relevant symmetries are the SU(3) symmetry which is broken by the couplings h, and the U(1) symmetry rotating all the chiral superfields together which is broken by h as well as by the gauge coupling and the anti-symmetric superpotential coupling. Non-zero gamma functions will induce complexified U(3) global symmetry transformations acting on the chiral superfields Φ^i, and it is easy to express those in terms of their action on gauge-invariant variables (since we know their global symmetry transformation). The scale-dependence of the various superpotential couplings is simply given by their transformation under these complexified global symmetry transformations. The scale-dependence of the gauge coupling is also proportional to a factor coming from this transformation, which is the numerator of the NSVZ formula for the beta function.

3. Supergravity analysis — methods

A deformation of a conformal theory which has a dual description in terms of string theory on AdS_5, such as $\mathcal{N} = 4$ SYM theory which is dual to type-IIB string theory on $AdS_5 \times S^5$, corresponds (for large $N, g_{\rm YM}^2 N$ where supergravity is a good approximation to string theory) to a solution of supergravity with appropriate boundary values for the fields [2, 3]. Marginal operators correspond to massless fields in supergravity, and for marginal deformations the corresponding massless SUGRA fields should approach a constant at the boundary of the AdS_5 space, while for relevant deformations they should decay like an appropriate power of the radial coordinate. Having an exactly marginal deformation means that none of the fields in the deformed solution should depend on the AdS coordinates, since the SO(4, 2) symmetry should remain exact. Thus, the fields can only obtain an angular dependence, and are given by some combination of S^5 spherical harmonics. Ideally, we would like to find exact solutions corresponding to the exactly marginal deformations described above, but we have not been able to do this so far. An alternative option, which we will pursue in this paper, is to try to construct the solutions in a perturbation series in the deformation around the $\mathcal{N} = 4$ fixed line. This approach only makes sense for exactly marginal deformations, since relevant or irrelevant deformations would not remain small throughout the AdS_5 space and the perturbation expansion would break down.

Our goal in this paper will be to show that the exactly marginal deformations which we analyzed in field theory in the previous section exist also in the AdS dual, in the supergravity limit. We begin in this section by examining how to analyze marginal deformations perturbatively in supergravity, and how to see if they are exactly marginal or not. In the first subsection we will analyze how to solve the supergravity equations of motion perturbatively in the deformation, and in the second subsection we will analyze how to solve the supersymmetry equations.

3.1 Perturbative expansion of the equations of motion

To leading order in the deformation away from the $\mathcal{N}=4$ theory, marginal deformations of this theory correspond to giving vacuum expectation values (VEVs) to the massless scalars of the SUGRA spectrum analyzed in [14]. The light scalars in the supergravity spectrum are depicted below in figure 1 (which is reproduced with permission from [14]). There is one complex marginal deformation corresponding to the zero mode (on the S^5) of the dilaton and the axion, which is obviously exactly marginal also in SUGRA, since the supergravity equations only depend on the derivatives of the dilaton (thus, this is in fact true for any supergravity solution in the supergravity approximation). This deformation corresponds to changing the gauge coupling τ in the field theory. There are complex marginal deformations in the **45** representation of $SU(4)_R$ corresponding to turning on a 2-form field (any combination of the NS-NS and R-R 2-form fields) on the S^5 proportional to the spherical harmonic $Y_{ab}^{[2,-]}$ (see [14] for definitions of the relevant spherical harmonics), and real marginal deformations in the **105** representation corresponding to changing a particular combination of the trace h_a^a of the metric on the S^5, the trace h_μ^μ of the AdS_5 metric, and the 5-form field, by an amount proportional to the spherical harmonic $Y^{[4]}$. Using the identification between the field theory operators and the supergravity fields [3] we

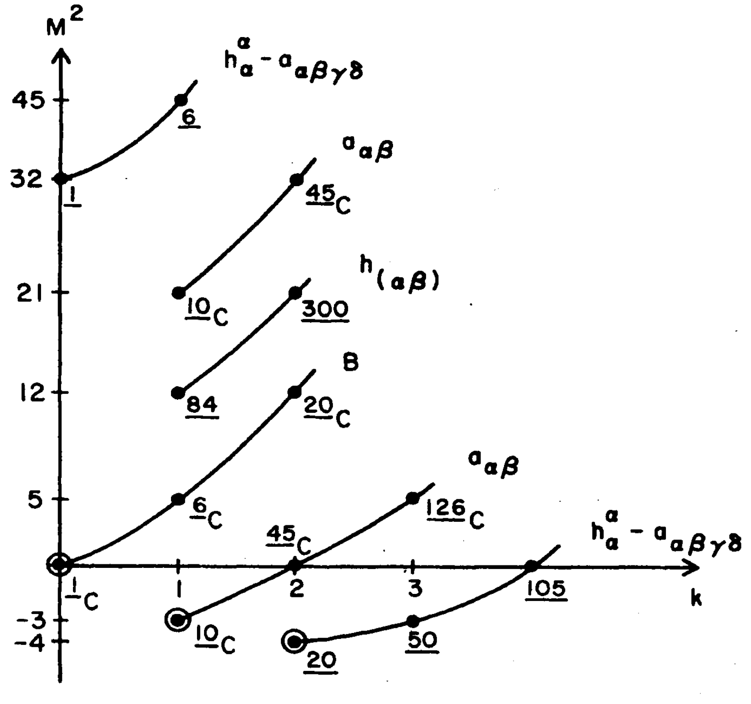

Figure 1: Spectrum of AdS scalars for IIB supergravity on $AdS_5 \times S^5$ [14].

find that the marginal deformations h_{ijk} described above involve fields in the **45**, to leading order in the deformation; at higher orders there will be a mixing of the various operators. In the $\mathcal{N}=4$ field theory the corresponding operators are descendants of the chiral primary operator $\mathrm{tr}(\varphi^{\{I}\varphi^J\varphi^{K\}})$ (where the φ's are the scalar fields in the $\mathcal{N}=4$ vector multiplet) constructed by acting on it with two SUSY generators; this gives rise to both Yukawa couplings and scalar potential terms (arising from the superpotential contributions to the SUSY variations).

There is a well-defined procedure for constructing a SUGRA solution, corresponding to an exactly marginal deformation, perturbatively in the deformation parameter. Obviously, such a deformation involves only fields which are scalars on the AdS_5. In [14], the SUGRA equations of motion were expanded to first order around the $AdS_5 \times S^5$ solution. The deviation of each of the AdS_5-scalar fields from its expectation value in the $AdS_5 \times S^5$ solution can be expanded in spherical harmonics, leading to a series of fields $\phi_{j,[k]}$, where $[k]$ labels the spherical harmonic and j labels the field. For each of these fields the linearized SUGRA equations of motion then take the form

$$\left(\Delta_{AdS} + m^2_{j,[k]}\right)\phi_{j,[k]} = 0\,. \tag{3.1}$$

The fields ϕ_j include the dilaton B ($B=(1+i\tau)/(1-i\tau)$), the NS-NS and R-R two-form fields $A^{1,2}_{ab}$ (indices $a,b,c,\ldots$ will be taken to run over the S^5 coordinates), the metric components h_{ab}, the trace of the metric h^a_a and the 4-form field A_{abcd}; the last two fields actually mix together, so the fields ϕ_j are in this case appropriate linear combinations of the two [14]. The spherical harmonics appearing in the expansion depend on the tensor type (with respect to S^5) of the field; for B, h^a_a and A_{abcd} they are the scalar spherical harmonics, for A_{ab} they are the anti-symmetric tensor spherical harmonics, and for h_{ab} they are symmetric tensor spherical harmonics. As described in [14], some of the fields in the spherical harmonic expansion may be gauged away by the symmetries of the SUGRA theory. In particular, the expansion of the tensor fields could include also derivatives of vector and scalar spherical harmonics, but these may be gauged away. The fields in the doubleton representation (corresponding to a free U(1) vector multiplet) may also be gauged away in the bulk of AdS_5. After these gauge fixings, the complete list of fields appearing in (3.1) is given in table III and figure 2 of [14], which we reproduced as figure 1 above.

To go beyond the linearized approximation, we need to expand the supergravity equations of motion to higher order in the fields $\phi_{j,[k]}$. We can do this systematically by expanding each of these fields in a power series in a deformation parameter h (which will be simply the superpotential coefficient discussed above), of the form $\phi_{j,[k]} = \sum_{i=1}^{\infty} h^i \phi^{(i)}_{j,[k]}$ (the generalization to having several expansion parameters is straightforward). Plugging this into the exact SUGRA equations of motion, we can expand them in a power series in h, and solve them order by order.

Since we are starting with an exact solution, the equations are satisfied to zeroth order in h. To first order in h, the equations are exactly the linearized equations (3.1) for the leading $\phi^{(1)}_{j,[k]}$ term. As expected, this means that if we want no AdS-dependence, we must turn on only massless fields at this order.

At the n'th order in this expansion, each equation will involve a term including $\phi^{(n)}_{j,[k]}$ with no other deformation fields, and other terms including lower order fields. The term involving $\phi^{(n)}_{j,[k]}$ will be exactly the linearized term in this field; thus, we will find equations of the form

$$\left(\Delta_{AdS} + m^2_{j,[k]}\right) \phi^{(n)}_{j,[k]} = f\left(\left\{\phi^{(1)}_{l,[m]}, \phi^{(2)}_{l,[m]}, \ldots, \phi^{(n-1)}_{l,[m]}\right\}\right), \tag{3.2}$$

for some function f involving various lower order fields on the right hand side. In principle, other spherical harmonics might appear in the expansion that have been gauged away at the leading order, so they do not correspond to fields $\phi_{j,[k]}$; however, we can change the gauge condition order by order to take account of such terms, and concentrate only on the equations of the form (3.2). There is a subtlety here, which is that in order to do this we may need to perform gauge transformations that do not vanish at infinity, and it is not clear if these should be allowed or not. If we do not allow these gauge transformations, we might find an obstruction from the "gauge modes" to the construction of a solution. Here we assume that all gauge transformations are allowed.

Since we are looking for AdS-independent solutions, only the mass term contributes on the left hand side, so for massive fields the equation simply determines the form of $\phi^{(n)}_{j,[k]}$. However, for massless fields, if the expression on the right hand side of the equation is non-zero, there is no AdS-independent solution. Thus, at each order in the perturbative expansion, we get one "obstruction" equation which the deformations must satisfy for each massless field in the SUGRA theory.[11] To check if a deformation is exactly marginal or not we should plug in the appropriate first order deformation, solve the equations order by order, and see if we encounter obstructions from the higher order massless field equations or not. If we find a non-zero term on the right-hand side for a massless field, we can still solve the equations but the fields will necessarily acquire a dependence on the radial coordinate, breaking the conformal invariance; this corresponds to cases with dimensional transmutation in the field theory.

As an example of this procedure let us assume that we are turning on a generic marginal deformation of $AdS_5 \times S^5$, and let us look at the simplest equation of motion of type-IIB supergravity,

$$D^M P_M = G_{ABC} G^{ABC}, \tag{3.3}$$

where $P_M = D_M B/(1-|B|^2)$ and G is some combination of the 3-form field strengths (we will use the conventions of [26] for type-IIB supergravity). In the linear approximation, this equation is simply $(\Delta_{AdS} + m^2_{[k]})B^{(1)}_{[k]} = 0$, where $B_{[k]}$ is the mode of the dilaton in the k'th spherical harmonic on the sphere, and $m^2_{[k]} \propto k(k+4)$, so we can only turn on the constant $k = 0$ mode. At second order, the equation will only depend on the fields in the **45** representation (recall that to leading order we can only turn on massless fields), so it will be of the form

$$\left(\Delta_{AdS} + m^2_{[k]}\right) B^{(2)}_{[k]} = \left(G^{(1)}_{abc}(G^{abc})^{(1)}\right)_{[k]}, \tag{3.4}$$

[11] The "obstruction" can be thought of as a single non-linear equation which has to be satisfied, and at each order in perturbation theory we encounter the n'th order terms in this equation.

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where on the right hand side we have a projection of G^2 on the appropriate spherical harmonic. Plugging in the explicit form of the deformation in the **45**, we find that the right hand side is a combination of second order ($x^{\{I}x^{J\}}$) and fourth order ($x^{\{I}x^Jx^Kx^{L\}}$) scalar spherical harmonics, with coefficients that are the Clebsch-Gordan coefficients for the $\mathbf{20'}$ and **105** representations (respectively) in the product $\mathbf{45} \times \mathbf{45}$. Thus, in this case we find no obstruction to solving the equations, since the singlet spherical harmonic (corresponding to the massless field $B_{[0]}$) did not appear on the right hand side. However, we find that generically we will need to turn on $B_{[2]}^{(2)}$ and $B_{[4]}^{(2)}$, which will be given by some quadratic function of the deformation parameters. It is not clear if turning on these constant (on AdS_5) values for some modes of the dilaton has any interpretation in the field theory — turning on the fields $B_{[2]}$ and $B_{[4]}$ with a particular radial dependence corresponds to deformations or VEVs in the $\mathcal{N}=4$ theory, but the interpretation of the constant mode of these fields is not clear. Presumably, it is related to operator mixing which occurs after the deformation. In section 4 we will compute such corrections to various fields in the case of the supersymmetric exactly marginal deformations, and it may be interesting to find a field theory interpretation for the results.

Similarly, we may analyze the other SUGRA equations. For the most general possible marginal deformation that we could turn on, at second order in the deformation we find no obstruction from the equations for the fields in the **1** and the **45** (the latter fact follows from simple group theory arguments), but there is a possible obstruction from the equation for the massless field in the **105**. The coefficient of this obstruction involves the Clebsch-Gordan coefficients for $\mathbf{105} \times \mathbf{105} \to \mathbf{105}$; this gives 105 quadratic equations in the coefficients of the first order deformation in the **105** that must be satisfied for the deformation to be exactly marginal. At third order, all the massless field equations are in principle non-trivial, so we find $105 + 45 * 2 + 1 * 2 = 197$ trilinear equations in the deformation parameters which must be satisfied, and so on at higher orders. Unfortunately, the resulting equations seem very complicated, so we cannot write them down explicitly.

The equations simplify to some extent if the deformation we turn on is only in the **45** representation; recall that (for some particular elements of the **45**) this is the deformation that is expected to be exactly marginal from the field theory arguments.[12] In this case, as described above, the second order equations may always be satisfied,[13] and they generically lead to the generation of $B^{(2)}$ in the $\mathbf{20'}, \mathbf{105}$ representations, of $h_{ab}^{(2)}$ in the $\mathbf{84}, \mathbf{729}$ representations and of $(h_a^a)^{(2)}$, $A_{abcd}^{(2)}$ in the $\mathbf{1}, \mathbf{20'}$ representations. These are the only representations that may appear from group theory arguments; apriori it is not clear if their

[12]Actually, the exactly marginal deformation which we analyzed in section 2 is supersymmetric, which means that it corresponds not just to an operator in the **45** but one also has to add in quadratic order in the deformation an operator in the **105**, corresponding to the scalar potential in the field theory. The coefficient of this operator is not uniquely determined by the deformation in the **45**, but rather it depends on which $\mathcal{N}=1$ supersymmetry we wish to preserve (this is clear since the scalar potential is proportional to $h\,\bar{h}$, but the product of $\mathbf{45} \times \overline{\mathbf{45}}$ does not contain the **105** representation). Since this additional deformation is at second order in h, this does not change the qualitative discussion of this section. A similar situation occurs with the supersymmetric mass term, which includes not only an operator in the **10** but also an operator in the $\mathbf{20'}$ representation [4].

[13]Up to possible obstructions coming from the gauge-fixing conditions.

coefficients in the actual SUGRA equations are all non-zero, but this seems likely since there is no reason for them to vanish. $A_{ab}^{(2)}$ is zero in this case (this is because the equations have a $\mathbb{Z}_2$ symmetry inverting the sign of A_{ab}). The first constraints on deformations in the **45** come from the equation for the massless field in $A_{ab}^{(3)}$; they involve the projection onto the **45** representation (spherical harmonic) of the product of the fields generated at the second order with the first order field in the **45** (or its conjugate in the $\overline{\mathbf{45}}$; the deformation we are turning on is of course hermitian, so it involves both fields). This leads to 45 trilinear equations in the coefficients of the deformation; apriori there is no obvious reason why these equations should have non-zero solutions, but it is not impossible from the supergravity point of view (and our field theory analysis of the previous section implies that it should indeed happen). At the fourth order we get 105 quartic equations from the equations for $(h_a^a)^{(4)}$, $A_{abcd}^{(4)}$, and so on. There is no obvious reason from the SUGRA point of view why all these equations should have any non-trivial solutions.

In section 4 we will solve some of these equations to third order and see that for specific choices of the deformation (corresponding to the exactly marginal deformations in the field theory) there are non-trivial solutions. However, first we will simplify the equations by using the fact (which we have not used so far) that we are interested in deformations that preserve some supersymmetry.

3.2 Perturbative expansion of the supersymmetry equations

In our analysis above we saw that general marginal deformations are in the **1**, **45** and **105** representations of SU(4), and each one leads to an obstruction as well (at every order in the perturbation expansion). For the particular marginal deformations discussed in section 2 which preserve supersymmetry, a $\mathrm{U}(1)_R$ subgroup of SU(4) (which is part of the superconformal algebra) is also preserved, so all the scalars which are charged under $\mathrm{U}(1)_R$ will automatically be zero, and only the fields neutral under $\mathrm{U}(1)_R$ participate. That limits the above set to **1**, $\mathbf{8}+\mathbf{10}$ and **27** representations of the global SU(3) group, arising from the $\mathbf{1},\mathbf{45}$ and **105** representations of SU(4), respectively. When we use later the full supersymmetry equations in addition to the $\mathrm{U}(1)_R$ invariance, we will find that the only supersymmetric deformation (in addition to the string coupling in the **1**) is the **10** mentioned in section 2.

As discussed above, from the field theory we would expect that the only obstruction should be in the **8** representation, of the form

$$[(h\bar{h})_{\mathbf{8}}]_i^j \equiv h_{ikl}\,\bar{h}^{jkl} - \frac{1}{3}(h_{klm}\,\bar{h}^{klm})\,\delta_i^j = 0\,, \tag{3.5}$$

since there should be a superconformal theory whenever this is satisfied. For the consistency of the AdS/CFT correspondence it must be the case that all the obstructions in the supergravity actually vanish when the quadratic expression $(h\bar{h})_{\mathbf{8}}$ vanishes, and we expect that this is the only case when all these obstructions vanish, but we do not know how to show this directly.

To make further progress we will use the additional information we have, which is that the deformation also preserves $\mathcal{N}=1$ superconformal symmetry. In addition to the bosonic

SO(4, 2) and $U(1)_R$ symmetries utilized above, this includes also fermionic generators, and we should be able to find a corresponding generalized Killing spinor ϵ. Again, we can analyze this in a perturbation expansion in h, and in an expansion in spherical harmonics. The expansion of ϵ will involve spinor spherical harmonics, which are described (with their relation to the bosonic spherical harmonics) in [14].

The equations we need to solve are the vanishing of the SUSY variation of the dilatino λ and the gravitino Ψ_M [13],

$$\begin{aligned}\delta\lambda &= \frac{i}{\kappa}\Gamma^M P_M \epsilon^* - \frac{i}{24}\not{G}\epsilon\,,\\ \delta\Psi_M &= \frac{1}{\kappa}\left(D_M - \frac{i}{2}Q_M\right)\epsilon + \frac{i}{480}\not{F}\Gamma_M\epsilon - \frac{1}{96}(2\not{G}\Gamma_M + \Gamma_M\not{G})\epsilon^*\,,\end{aligned} \tag{3.6}$$

where $\not{G} \equiv \Gamma^{MNP}G_{MNP}$, $\not{F} \equiv \Gamma^{M_1M_2M_3M_4M_5}F_{M_1M_2M_3M_4M_5}$, and ϵ is a Weyl spinor, obeying $\Gamma_{11}\epsilon = -\epsilon$ (λ and Ψ_M are also Weyl spinors with $\Gamma_{11}\lambda = \lambda$, $\Gamma_{11}\Psi_M = -\Psi_M$). The notations of this equation are described in detail in section 4.1 below. We are looking for solutions which preserve superconformal invariance. Note that any solution to (3.6) is also a solution to the supergravity equations of motion, so solving these equations is enough for our purposes.

Since we are looking for a superconformally invariant solution it is convenient to decompose the spinor in a way which takes this into account. One way to do this, described in [14], is to partition the 10 directions into $5+5$, and write the 10 dimensional gamma matrices in terms of two groups of 5 dimensional gamma matrices. This decomposition does not seem to be very convenient for our purposes, and instead we will use here a different decomposition, as $10 = 4+6$, writing the gamma matrices as the outer product of SO(3, 1) gamma matrices (which are 4×4) and SO(6) gamma matrices (which are 8×8). This is the same decomposition appearing in [12] and it readily allows the description of flows when conformal invariance is violated. The 10 dimensional chirality matrix Γ_{11} may be written as a product $\Gamma^{11} = -\Gamma^4 \otimes \Gamma^7$ of four dimensional and six dimensional chirality matrices, so that the spinor ϵ will have components with Γ^4 and Γ^7 eigenvalues both positive or both negative.

In the $AdS_5 \times S^5$ background, the zeroth order dilatino equation is trivially solved, and the zeroth order gravitino variation equation (we will write the $AdS_5 \times S^5$ background as $ds^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}dx^m dx^m$ where $\mu, \nu = 0, \ldots, 3$, $m, n = 4, \ldots, 9$, $Z = R_0^4/r^4$ and $r^2 = x^m x^m$) says that [12]

$$\begin{aligned}0 = \kappa\delta\Psi_\mu &= \partial_\mu\epsilon + \frac{1}{2}\Gamma_\mu\Gamma^m\frac{\partial_m r}{r}(1-\Gamma^4)\epsilon\,,\\ 0 = \kappa\delta\Psi_m &= \partial_m\epsilon - \frac{1}{2}\epsilon\frac{\partial_m r}{r} + \frac{1}{2}\Gamma^n\frac{\partial_n r}{r}\Gamma_m(1-\Gamma^4)\epsilon\,.\end{aligned} \tag{3.7}$$

The solution to this is

$$\epsilon = \left(\frac{R_0}{r}\right)^{-1/2}\zeta\otimes\chi\,, \tag{3.8}$$

where ζ is a constant SO(3, 1) spinor with positive Γ^4 chirality, and χ is a constant SO(6) spinor with positive Γ^7 chirality. There are four possible constant SO(6) spinors so we get four $d = 4$ supersymmetries. After we deform only one of these will be preserved, so we choose the zeroth order SUSY parameter ϵ to be of the form (3.8) where $\chi = \chi_0$ is a particular one of the four possible constant SO(6) spinors.

The bosonic field equations have an $\mathrm{SL}(2,\mathbb{R})$ symmetry, with a $\mathbb{Z}_2$ subgroup that acts by changing the sign of the 3-form field G. Thus, it is clear that when we turn on a first order deformation involving purely the 3-form field G, as we are doing in the case of the SUSY marginal deformations, then in the perturbation expansion G will only get contributions in odd orders in the deformation parameter h, while the other SUGRA fields (the metric, dilaton and 5-form) will only get contributions at even orders in h. We can choose the matrix Γ^4 to be purely imaginary, so that ϵ^* has an opposite Γ^4 eigenvalue from ϵ. The vanishing of (3.6) then implies that at even orders in the deformation ϵ has a positive Γ^4 eigenvalue, while at odd orders ϵ has a negative Γ^4 eigenvalue (and ϵ^* has a positive Γ^4 eigenvalue) [12].

As in the previous subsection, we can expand the SUSY variation equations (3.6) in a power series in h, and at higher orders we may encounter obstructions. In the SUSY variation equations the obstructions are fermionic. We will linearize the equations (3.6) in full generality in section 4.4 below, but for now let us discuss a specific type of obstruction that could occur (additional obstructions will be discussed in the next section). Let us analyze the second equation of (3.6) in the case that M is an S^5 coordinate. Suppose that we knew all the supergravity fields up to order n in the deformation parameter h, and that we know ϵ up to order $n-1$. Then, this equation gives an equation for ϵ at n'th order, analogous to the equation (3.2) that we analyzed above, of the schematic form

$$(\partial_a + (A_{[k]})_a^b \Gamma_b)\epsilon_{[k]}^{(n)} = g_a\left(\left\{\phi_{l,[m]}^{(1)}, \cdots, \phi_{l,[m]}^{(n)}, \epsilon^{(0)}, \cdots, \epsilon^{(n-1)}\right\}\right), \tag{3.9}$$

for some matrix $A_{[k]}$. For most spherical harmonics this will just give a linear equation determining $\epsilon_{[k]}^{(n)}$. As above, this will not happen only for the case when the differential operator $(\partial_a + (A_{[k]})_a^b \Gamma_b)$ has zero eigenvalues, and then we will get an "obstruction" equation for the vanishing of the corresponding spherical harmonic on the right-hand side, which must be satisfied (as we found for the massless fields in the previous subsection). By studying the case of $n = 0$ it is clear that one case in which this will happen is for ϵ in the **4** representation of SU(4), since this is the only solution to the spinor equations in the original $AdS_5 \times S^5$ theory. Thus, at each order in perturbation theory we get a possible obstruction from this equation in the **4** of SU(4) (actually this obstruction will only appear at even orders). In fact, for the deformations we are interested in which preserve a $\mathrm{U}(1)_R$ in SU(4), the obstruction can only be in the singlet appearing in the decomposition $\mathbf{4} \to \mathbf{1}+\mathbf{3}$. Thus, for the case we are interested in, the second equation with an S^5 index gives a singlet obstruction at every even order of perturbation theory, and otherwise determines $\epsilon^{(n)}$ given the other fields at order n. We will see below that there are additional obstructions at odd orders.

In this paper we will not analyze the general structure of the obstructions, but rather we will just expand the equations and identify any obstructions when they show up.

4. Supergravity — perturbative computations

Our objective is to find deformations of type-IIB supergravity on $AdS_5 \times S^5$ which preserve the SO(4, 2) isometries of AdS_5 and 8 real supercharges (out of 32).

In section 2 we saw that $\mathcal{N} = 4$ SU(N) SYM theory (with $N \geq 3$) has 10 complex[14] super-marginal operators (marginal operators preserving $\mathcal{N} = 1$ supersymmetry) besides the complex gauge coupling. Two out of these are exactly marginal, given by

$$(h\bar{h})_{\mathbf{8}} = 0\,, \tag{4.1}$$

while the others fail at third order to be exactly marginal (they are marginally irrelevant). In section 3 we discussed some general features of the perturbative expansion in supergravity.

In the current section we present this computation in full detail, and find the following results:

- The full supergravity solution at second order, which includes the metric, the warp factor, the dilaton (which was already found in [12]) and the Killing spinor.
- A specific obstruction is shown to eliminate the non-exactly-marginal deformations, and we compute the third order beta function arising in this case.

However, so far we were not able to prove that there is no obstruction to extending the solutions obeying (4.1) to all orders.

4.1 Equations and variables

We are looking for backgrounds preserving SUSY, and hence our basic equations are the vanishing of the supersymmetry variation of the fermionic fields in type-IIB supergravity. These, together with an ansatz with SO(4, 2) isometry, will guarantee that our solutions will solve the type-IIB supergravity equations of motion and will have the full superconformal symmetry. The equations are of the form [13]:

$$\begin{aligned}
\delta\Psi_M &= \frac{1}{\kappa}\left(\hat{D}_M - \frac{i}{2}Q_M\right)\epsilon + \frac{i}{480}\hat{\not{F}}\,\Gamma_M\,\epsilon - \frac{1}{96}(2\hat{\not{G}}\,\Gamma_M + \Gamma_M\,\hat{\not{G}})\epsilon^* + \text{fermions},\\
\delta\lambda &= \frac{i}{\kappa}\hat{\not{P}}\,\epsilon^* - \frac{i}{24}\hat{\not{G}}\,\epsilon\,.
\end{aligned} \tag{4.2}$$

The hatted quantities $\hat{G}, \hat{F}, \hat{P}, \hat{D}$ denote supercovariant versions of the unhatted quantities which differ from them only by fermionic terms. After setting the fermions to zero and setting $\kappa = 1$ (GR units) we get

$$\begin{aligned}
\delta\Psi_M &= \left(D_M - \frac{i}{2}Q_M\right)\epsilon + \frac{i}{480}\not{F}\,\Gamma_M\,\epsilon - \frac{1}{96}(2\not{G}\,\Gamma_M + \Gamma_M\,\not{G})\epsilon^*,\\
\delta\lambda &= i\not{P}\,\epsilon^* - \frac{i}{24}\not{G}\,\epsilon\,,
\end{aligned} \tag{4.3}$$

[14] 11 at zero coupling, where the anti-symmetric superpotential is also marginal.

where

$$
\begin{aligned}
D_M &= \partial_M + \frac{1}{4}\omega_M^{NP}\,\Gamma_{NP}\,,\\
G &= \frac{(F_3 - B\,F_3^*)}{\sqrt{1-|B|^2}}\,, \qquad F_3 = dA_2\,,\\
\not{G} &= \Gamma^{PQR}G_{PQR}\,,\\
F &= dA_4 - \frac{1}{8}\,\mathrm{Im}(A_2\wedge F_3^*)\,, \qquad F = +*F\,,\\
\not{F} &= \Gamma^{MNPQR}F_{MNPQR}\,,\\
Q_M &= \mathrm{Im}\,\frac{(B\,\partial_M B^*)}{(1-|B|^2)}\,,\\
P_M &= \frac{\partial_M B}{(1-|B|^2)}\,.
\end{aligned} \tag{4.4}
$$

Our notations are: ω is the spin connection; B is the complex scalar related to the complexified string coupling $\tau \equiv \frac{i}{g_s} + \chi$ by a conformal mapping of the upper half plane to the unit disk, $B = (1+i\tau)/(1-i\tau)$; G (or F_3) is the complex 3-form field strength related to the complex 2-form potential A_2 which combines the NS-NS and the RR two forms; F is the self-dual 5-form field strength; Ψ_M is the complex gravitino; λ is the complex dilatino; and ϵ is the complex Killing spinor (including two Majorana spinors).

The Clifford algebra is defined as usual by $\{\Gamma_M, \Gamma_N\} = 2g_{MN}$. The Γ_{11} chiralities of the spinors are $+,-,-$ for $\lambda, \Psi_M, \epsilon$ respectively ($\Gamma_{11} = \Gamma^0 \dots \Gamma^9$, $\Gamma_{11}^2 = +1$), and the signature of the metric is $(+,-,\dots,-)$. Products $\Gamma^{M_1\cdots M_n}$ of gamma matrices are defined to be anti-symmetrized and divided by the order of the permutation group, such that when all indices are different $\Gamma^{M_1\cdots M_n} = \Gamma^{M_1}\cdots\Gamma^{M_n}$. These equations are written in a Majorana basis. In order to write them in any other basis we need to replace $\epsilon^* \to C\epsilon^*$ where C is the charge conjugation matrix in that basis.

4.2 The ansatz

The metric. The $AdS_5 \times S^5$ solution is given by

$$
\begin{aligned}
ds^2 &= {R_0}^2\,ds^2_{AdS_5} - {R_0}^2\,ds^2_{S^5} = \frac{r^2}{{R_0}^2}dx^\mu dx_\mu - \frac{{R_0}^2}{r^2}dx^\alpha dx_\alpha\,,\\
F &= \pm\frac{1}{R_0}\left(\left(\frac{r}{R_0}\right)^3 dx^0\,dx^1\,dx^2\,dx^3\,dr + \left(\frac{R_0}{r}\right)^5 dx^4\cdots dx^9\cdot dr\right),\\
B &= B_0\,,
\end{aligned} \tag{4.5}
$$

with all other fields vanishing. In this familiar form the AdS_5 is composed out of $\mathbb{R}^4$ together with the radial coordinate of $\mathbb{R}^6$, and hence we refer to it as the 4-6 split (compared to a possible notation with a 5-5 split). The indices $\mu,\nu,\dots$ run over $0,\dots,3$, while $\alpha,\beta,\dots$ run over $4,\dots,9$, M,N run over the whole range[15] $0,\dots,9$ and $r^2 = x_\alpha x^\alpha$ is the radial coordinate in $\mathbb{R}^6$. The free $\pm$ sign allows to choose between the D3 and anti-D3 solution

[15] Later we will introduce also lower case Roman indices $i,j,\dots$ to denote holomorphic indices in $\mathbb{R}^6$.

(later we choose $-$). The $\cdot$ stands for contraction of the 1-form dr with the 6-form on its left through the flat $\mathbb{R}^6$ metric, namely $F_{0123\alpha} = -r^2\, x^\alpha/R_0^4$, $F_{\alpha_1\ldots\alpha_5} = -\epsilon_{\alpha_1\ldots\alpha_6}\, x^{\alpha_6} R_0^4/r^6$. This solution contains one free dimensionful parameter (as always in classical GR after setting $G = c = 1$) which is R_0, the size of AdS and the sphere. From now on we will set $R_0 = 1$, and it can always be restored by dimensional analysis (we will restore the factors of R_0 in the next section when comparing to field theory). The complex coupling B can have any constant value, and in supergravity all of them are related by the $\mathrm{SL}(2,\mathbb{R})$ symmetry (this symmetry is broken down to $\mathrm{SL}(2,\mathbb{Z})$ in string theory). We will use this freedom to set $B_0 = 0$ (or $\tau = i$) at zeroth order in the deformation.

The most general ansatz for the form of the metric after the deformation, which preserves the AdS_5 isometries, is a warped fibration of AdS_5 over a deformed $S^{5'}$, namely

$$\widetilde{ds}^2 = \exp(2\rho)\, ds^2_{AdS_5} - ds^2_{S^{5'}}. \tag{4.6}$$

Written in a 4-6 split of coordinates our ansatz for the 10d metric is

$$\begin{aligned}\widetilde{ds}_{10}^2 &= e^{2\rho}\left[r^2\,\eta_{\mu\nu}\,dx^\mu dx^\nu - \frac{1}{r^2}\,ds^2\right]\\ &= f_{||}^2\,\eta_{\mu\nu}\,dx^\mu dx^\nu - f_\perp^2\,ds^2,\end{aligned} \tag{4.7}$$

where $f_{||} \equiv r\,e^\rho$, $f_\perp \equiv e^\rho/r$, and we denote the 10d metric quantities with a tilde and the 6d metric without one, $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. At zeroth order ds^2 is the flat metric on $\mathbb{R}^6$, $ds_0^2 = dr^2 + r^2\, ds^2_{S^5}$, and $\rho = 0$.

The ansatz (4.7) has manifest 4d Lorentz invariance. In order to guarantee the invariance under all the AdS_5 isometries we require that:

$$\begin{aligned}&ds^2(x^\alpha) \text{ is homogeneous of degree } 2\,,\\ &\rho(x^\alpha) \text{ is homogeneous of degree zero}\,,\\ &g_{\alpha\beta}(x^\gamma)\, x^\beta = x_\alpha \equiv \delta_{\alpha\beta}\, x^\beta\,.\end{aligned} \tag{4.8}$$

The ansatz should also have a U(1) isometry (corresponding to the $\mathrm{U}(1)_R$ in the $\mathcal{N} = 1$ superconformal algebra) which we discuss later after introducing complex coordinates.

Let us compute the connection $\tilde\omega$ for the metric (4.7). The frame of 1-forms is

$$\begin{aligned}\widetilde{\Omega}^{\hat\mu} &= f_{||}\, dx^\mu,\\ \widetilde{\Omega}^{\hat\alpha} &= f_\perp\, \Omega^{\hat\alpha} = f_\perp\, \Omega^{\hat\alpha}{}_\beta\, dx^\beta\,,\end{aligned} \tag{4.9}$$

where hatted indices are in orthonormal frames, and $\widetilde{\Omega}$, Ω are the frames of 1-forms for $\tilde g$, g respectively. A frame of 1-forms is defined by the orthogonality relation $g = \Omega^{\hat\alpha}\,\Omega^{\hat\beta}\,\eta_{\hat\alpha\hat\beta}$ where $\eta_{\hat\alpha\hat\beta}$ is the standard flat metric. We will also use the frame of tangent vectors, $e_{\hat\alpha}$ (and $\widetilde{e}_{\hat\alpha}$ for $\tilde g$), defined by $dx^\beta = e^\beta_{\hat\gamma}\,\Omega^{\hat\gamma}$ namely $e^\beta{}_{\hat\gamma}$ is the inverse of $\Omega_\beta{}^{\hat\gamma}$.

To find the connection we use Cartan's structural equations

$$\begin{aligned}d\Omega^{\hat M} + \omega^{\hat M}_{\ \hat N}\,\Omega^{\hat N} &= 0\,,\\ \omega_{\hat M\hat N} + \omega_{\hat N\hat M} = dg_{\hat M\hat N} &= 0\,.\end{aligned} \tag{4.10}$$

In the case of an $\mathbb{R}^4$ index we have

$$d\widetilde{\Omega}^{\hat{\mu}} = df_{\|}\, dx^\mu = \partial_\alpha f_{\|}\, dx^\alpha dx^\mu = \frac{\partial_\alpha f_{\|}}{f_\perp}\, e^\alpha_{\hat{\beta}}\, \widetilde{\Omega}^{\hat{\beta}}\, dx^\mu\,, \tag{4.11}$$

and the solution is

$$\begin{aligned}
\widetilde{\omega}^{\hat{\mu}}{}_{\hat{\nu}} &= 0\,,\\
\widetilde{\omega}^{\hat{\mu}}{}_{\hat{\beta}} &= +\frac{\partial_\alpha f_{\|}}{f_\perp}\, e^\alpha{}_{\hat{\beta}}\, dx^\mu\,,\\
\widetilde{\omega}^{\hat{\beta}}{}_{\hat{\mu}} &= -\eta^{\hat{\beta}\hat{\gamma}}\, \eta_{\hat{\mu}\hat{\nu}}\, \omega^{\hat{\nu}}{}_{\hat{\gamma}}\,.
\end{aligned} \tag{4.12}$$

The equations for an $\mathbb{R}^6$ index are

$$\begin{aligned}
d\widetilde{\Omega}^{\hat{\alpha}} &= df_\perp\, \Omega^{\hat{\alpha}} + f_\perp\, d\Omega^{\hat{\alpha}}\\
&= -\partial_\beta f_\perp\, \Omega^{\hat{\alpha}}\, dx^\beta - f_\perp\, \omega^{\hat{\alpha}}{}_{\hat{\beta}}\, \Omega^{\hat{\beta}} = -f_\perp^{-1}\, \partial_\beta f_\perp\, \Omega^{\hat{\alpha}}\, e^\beta{}_{\hat{\gamma}}\, \widetilde{\Omega}^{\hat{\gamma}} - \omega^{\hat{\alpha}}{}_{\hat{\beta}}\, \widetilde{\Omega}^{\hat{\alpha}},
\end{aligned} \tag{4.13}$$

and the solution is

$$\widetilde{\omega}^{\hat{\alpha}}{}_{\hat{\beta}} = f_\perp^{-1}\, \partial_\gamma f_\perp (\Omega^{\hat{\alpha}}\, e^\gamma{}_{\hat{\beta}} - \Omega_{\hat{\beta}}\, e^{\gamma\hat{\alpha}}) + \omega^{\hat{\alpha}}{}_{\hat{\beta}}\,. \tag{4.14}$$

In order to evaluate the equations (4.2) we need the expression for the spin connection. Using our results above, in the $\mathbb{R}^4$ directions:

$$\begin{aligned}
\frac{1}{4}\, \widetilde{\omega}_{\mu\hat{M}\hat{N}}\, \Gamma^{\hat{M}\hat{N}} &= \frac{1}{2}\, \widetilde{\omega}_{\mu\hat{\mu}\hat{\beta}}\, \Gamma^{\hat{\mu}\hat{\beta}}\\
&= \frac{1}{2}\, \frac{\partial_\alpha f_{\|}}{f_\perp}\, e^\alpha_{\hat{\beta}}\, \Gamma_{\hat{\mu}}{}^{\hat{\beta}}\\
&= \frac{1}{2}(r^2\, \partial_\alpha \rho + x_\alpha)\Gamma_{\hat{\mu}}{}^\alpha\\
&= \frac{1}{2} r \frac{\widetilde{\Gamma}_{\hat{\mu}}(r\, \partial\!\!\!/ \rho + \Gamma_r)}{f_{\|}}
\end{aligned} \tag{4.15}$$

where the various gamma matrices are defined as usual by $\{\Gamma_M, \Gamma_N\} = 2g_{MN}$, $\{\Gamma_{\hat{M}}, \Gamma_{\hat{N}}\} = 2\eta_{\hat{M}\hat{N}}$, $\{\widetilde{\Gamma}_\alpha, \widetilde{\Gamma}_\beta\} = 2\widetilde{g}_{\alpha\beta}$. In the orthonormal frame $\Gamma = \widetilde{\Gamma}$. We define $\Gamma_r = x^\alpha\, \Gamma_\alpha / r$, which satisfies $(\Gamma_r)^2 = +1$.

Equation (4.15) suggests to define $\phi\!\!\!/_4$ by

$$\begin{aligned}
\frac{1}{4}\, \widetilde{\omega}_{\mu\hat{M}\hat{N}}\, \Gamma^{\hat{M}\hat{N}} &= \widetilde{\Gamma}_\mu\, \phi\!\!\!/_4\,,\\
\phi\!\!\!/_4 &= \frac{1}{2}\, e^{-\rho}(r\, \partial\!\!\!/ \rho + \Gamma_r)\,.
\end{aligned} \tag{4.16}$$

In the $\mathbb{R}^6$ directions the spin connection is

$$\frac{1}{4}\, \widetilde{\omega}_{\alpha\hat{M}\hat{N}}\, \widetilde{\Gamma}^{\hat{M}\hat{N}} = \frac{1}{4}\, \omega_{\alpha\hat{M}\hat{N}}\, \Gamma^{\hat{M}\hat{N}} + \frac{1}{4}\left(-\partial_\beta \rho + \frac{x_\beta}{r^2}\right)(\Gamma^\beta\, \Gamma_\alpha - \Gamma_\alpha\, \Gamma^\beta)\,. \tag{4.17}$$

Other fields. We will take the 3-form field to preserve the SO(4, 2) invariance, namely only $G_{\alpha\beta\gamma}$ will be non-zero, and it will be homogeneous of degree (-3) and obey $x^\alpha G_{\alpha\beta\gamma}=0$.

Given our ansatz for the metric, the form of the 5-form field strength F is uniquely determined by the equations $F = *F$, $dF = -4iG \wedge G^* = 0$ (the last equation follows from the fact that we only turn on G with indices in the five compact directions), up to an overall constant f_5:

$$F = -f_5 e^{-5\rho} \left(\widetilde{\Omega}^{\hat{0}}\, \widetilde{\Omega}^{\hat{1}}\, \widetilde{\Omega}^{\hat{2}}\, \widetilde{\Omega}^{\hat{3}}\, \widetilde{\Omega}^{\hat{r}} + \widetilde{\Omega}^{\hat{4}}\, \widetilde{\Omega}^{\hat{5}}\, \widetilde{\Omega}^{\hat{6}}\, \widetilde{\Omega}^{\hat{7}}\, \widetilde{\Omega}^{\hat{8}}\, \widetilde{\Omega}^{\hat{9}} \cdot \widetilde{\Omega}^{\hat{r}} \right). \tag{4.18}$$

The $\cdot$ stands for contraction of the 1-form $\Omega^{\hat{r}} \equiv \Omega^{\hat{\alpha}}\, x_\alpha / r$ with the 6-form on its left through the metric, and the overall $(-)$ sign is chosen so that the SUSY variation equations (through equation (4.19)) match those of [12] for $f_5 = 1$. The constant f_5 is fixed by the normalization condition, which in our conventions (including $R_0 = \kappa = 1$) is given by $-\int_{S^5} F = 1$. The following expressions will be needed for the equations:

$$\frac{i}{480} \not{F} = \frac{f_5}{4} e^{-5\rho}\, \Gamma_r\, (\Gamma^7 - \Gamma^4) \tag{4.19}$$

$$1 = -\int_{\widetilde{S^5}} F = f_5 \int_{S^5} \sqrt{\det(g_{\alpha\beta})}\, d^5x\,, \tag{4.20}$$

where we used equations (4.8),(4.18), and $g_{\hat{r}\hat{r}} = 1$.

We use the chirality matrix definitions[16] $\Gamma^7 = i\, \epsilon_{\alpha 1 \ldots \alpha 6}\, \Gamma^{\alpha 1} \cdots \Gamma^{\alpha 6}/(6!) = i\, \Gamma^{4 \ldots 9}$, $\Gamma^4 = +i\, \Gamma^{0123}$, which satisfy $\Gamma^7 = -\Gamma^4\, \Gamma_{11}$ and thus conform with the sign conventions of [12]. Since $\Gamma_{11}\, \epsilon = -\epsilon$ we have $\Gamma^7\, \epsilon = \Gamma^4\, \epsilon$.

We take the ansatz for the Killing spinor to be

$$\epsilon = f_\perp^{-0.5} \chi = f_\perp^{-0.5} (\zeta\, \chi^{(e)} + \zeta^*\, \chi^{(o)})\,, \tag{4.21}$$

where we explicitly factored out the dependence of ϵ on the AdS_5 coordinates. This form uses the 4-6 split and the fact that the $\mathbb{R}^4$ spinor part factors out of the equations, except for complex conjugation, to decompose χ into the product of an $\mathbb{R}^4$ spinor and an $\mathbb{R}^6$ spinor. ζ is an arbitrary constant positive chirality $\mathbb{R}^4$ spinor $\Gamma^4\, \zeta = +\zeta$, $\Gamma^4\, \zeta^* = -\zeta^*$, and the chiralities of the $\mathbb{R}^6$ spinors are $\Gamma^7\, \chi^{(e)} = +\chi^{(e)}$, $\Gamma^7\, \chi^{(o)} = -\chi^{(o)}$ where the superscript e/o stands for even/odd chirality (this will turn out to agree also with the order in perturbation theory).

4.3 Substitution of the ansatz into the equations

Using the 4d Lorentz symmetry we can reduce all the equations to 6d, and since at zeroth order the untilded metric ds^2 is flat $\mathbb{R}^6$, we choose to work always in the untilded frame (with positive $(+, \ldots, +)$ metric). Due to 4d Lorentz symmetry the equation for the gravitino components along $\mathbb{R}^4$ can be simplified into an algebraic equation for the spinor. Let us collect the terms needed for this equation:

$$D_\mu - \frac{i}{2} Q_\mu = \partial_\mu + \frac{1}{4} \widetilde{\omega}_{\mu \hat{M} \hat{N}}\, \Gamma^{\hat{M}\hat{N}} = \widetilde{\Gamma}_\mu \not{\partial}_4,$$

[16] No confusion should arise with the Γ matrix associated with x^7 as we will soon restrict ourselves to 6d.

$$\begin{aligned} \frac{i}{480} \not{F} \widetilde{\Gamma}_\mu &= -\widetilde{\Gamma}_\mu \frac{f_5}{4} e^{-5\rho} \Gamma_r (\Gamma^7 + \Gamma^4), \\ -2 \not{G} \widetilde{\Gamma}_\mu - \widetilde{\Gamma}_\mu \not{G} &= +\widetilde{\Gamma}_\mu \not{G}, \end{aligned} \tag{4.22}$$

where we used (4.16) for ω_μ and the fact that $\partial_\mu = 0$ for all fields in the ansatz (including $Q_\mu = 0$) in the first line, (4.19) in the second, and $\not{G} \widetilde{\Gamma}_\mu + \widetilde{\Gamma}_\mu \not{G} = 0$ in the third. Assembling them, using the ansatz for ϵ (4.21), and the rescaled variable $\delta\psi \equiv f_\perp^{1.5} \widetilde{\Gamma}^\mu \delta\Psi_\mu/4$ we obtain

$$\delta\psi = \left[\frac{1}{2r} \Gamma_r (1 - f_5 e^{-4\rho} \Gamma^7) + \frac{1}{2} \not{\partial}\rho \right] \chi + \frac{e^\rho}{96\, r} \not{G} \chi^* . \tag{4.23}$$

Next, we collect the terms required for the gravitino equation with an $\mathbb{R}^6$ index:

$$\tilde{D}_\alpha = D_\alpha + \frac{1}{4} \left(-\partial_\beta \rho + \frac{x_\beta}{r^2} \right) (\Gamma^\beta \Gamma_\alpha - \Gamma_\alpha \Gamma^\beta) , \tag{4.24}$$

$$\begin{aligned} f_\perp^{0.5} (\partial_\alpha f_\perp^{-0.5}) &= \partial_\alpha - \frac{\partial_\alpha f_\perp}{2 f_\perp} = \partial_\alpha + \frac{1}{2} \left(-\partial_\alpha \rho + \frac{x_\alpha}{r^2} \right) \Rightarrow \\ \Rightarrow f_\perp^{0.5} \tilde{D}_\alpha f_\perp^{-0.5} &= D_\alpha + \frac{1}{2} \left(-\not{\partial}\rho + \frac{\Gamma_r}{r} \right) \Gamma_\alpha , \end{aligned} \tag{4.25}$$

$$\frac{i}{480} \not{F} \widetilde{\Gamma}_\alpha = -\frac{f_5}{4\, r} e^{-4\rho} \Gamma_r \Gamma_\alpha (\Gamma^7 + \Gamma^4) . \tag{4.26}$$

Assembling them, using the ansatz for ϵ (4.21), and the rescaled variable $\delta\psi_\alpha \equiv f_\perp^{0.5} \delta\Psi_\alpha$, we get

$$\begin{aligned} \delta\psi_\alpha = & \left[D_\alpha - \frac{i}{2} Q_\alpha + \frac{1}{2\, r} \Gamma_r \Gamma_\alpha (1 - f_5 e^{-4\rho} \Gamma^7) - \frac{1}{2} \not{\partial}\rho \Gamma_\alpha \right] \chi - \\ & - \frac{e^\rho}{96\, r} (2 \not{G} \Gamma_\alpha + \Gamma_\alpha \not{G}) \chi^* . \end{aligned} \tag{4.27}$$

Once "traced", the vector-spinor equation (4.27) gives

$$\Gamma^\alpha \delta\psi_\alpha = \left[\not{D} - \frac{i}{2} \not{Q} - \left(\frac{2}{r} \right) \Gamma_r (1 - f_5 e^{-4\rho} \Gamma^7) + 2 \not{\partial}\rho \right] \chi - 6 \frac{e^\rho}{96\, r} \not{G} \chi^* , \tag{4.28}$$

where we used $\Gamma^\alpha \Gamma^\beta \Gamma_\alpha = -4 \Gamma^\beta$ and $\Gamma^\alpha \not{G} \Gamma_\alpha = (-3 \cdot 2 + 6) \not{G} = 0$. This equation can be combined with the spinor equation to create a linear combination $\delta\tilde{\psi} \equiv \Gamma^\alpha \delta\psi_\alpha + 6\, \delta\psi$ where all the $\not{G}\chi^*$ terms are absent. This combination will play an important role later, and it is given by

$$\boxed{\delta\tilde{\psi} = \Gamma^\alpha \delta\psi_\alpha + 6\, \delta\psi = \left[\not{D} - \frac{i}{2} \not{Q} + (1/r) \Gamma_r (1 - f_5 e^{-4\rho} \Gamma^7) + 5 \not{\partial}\rho \right] \chi = 0 .} \tag{4.29}$$

After one additional rescaling $\lambda = -i\, f_\perp^{0.5} \lambda$, we may list the full equations (4.2) in our ansatz:

$$\boxed{\begin{aligned} 0 = \delta\psi_\alpha &= \left[D_\alpha - \frac{i}{2}\,Q_\alpha + \frac{1}{2\,r}\Gamma_r\,\Gamma_\alpha\,(1 - f_5\,e^{-4\rho}\,\Gamma^7) - \frac{1}{2}\,\partial\!\!\!/\rho\,\Gamma_\alpha\right] - \chi \\ &\quad - \frac{e^\rho}{96\,r}(2\,G\!\!\!/\,\Gamma_\alpha + \Gamma_\alpha\,G\!\!\!/)\,\chi^*\,, \\ 0 = \delta\psi &= \left[\frac{1}{2r}\,\Gamma_r\,(1 - f_5\,e^{-4\rho}\,\Gamma^7) + \frac{1}{2}\,\partial\!\!\!/\rho\right]\chi + \frac{e^\rho}{96\,r}G\!\!\!/\chi^*\,, \\ 0 = \delta\lambda &= P\!\!\!/\chi^* - \frac{4}{96}\,G\!\!\!/\chi\,. \end{aligned}} \tag{4.30}$$

When we use the 4-6 split of the spinor, each equation above actually decomposes in two according to the ζ and ζ^* components in the ansatz (4.21).

The Q_α term will not be needed for our calculations since it will turn out to be quartic in the perturbation parameters h: $Q \sim B\,B^*$, $B \sim h^2$, and we will only work to third order in h.

At zeroth order the equations reduce to[17]

$$\begin{aligned} (\delta\psi_\alpha)^{(0)} &= \partial_\alpha\,\chi^{(0)} + \Gamma_r\,\Gamma_\alpha\,\frac{1-\Gamma^7}{2r}\,\chi^{(0)} = 0\,, \\ (\delta\psi)_0 &= \Gamma_r\,\frac{1-\Gamma^7}{2r}\,\chi^{(0)} = 0\,, \end{aligned} \tag{4.31}$$

from which we see that all 4 constant, positive chirality $\mathbb{R}^6$ spinors $\Gamma^7\chi^{(0)} = +\chi^{(0)}$ are solutions. Since we can combine each of these spinors with 2 independent positive chirality $\mathbb{R}^4$ spinors, we find 8 complex spinors, or 16 real supercharges (we discuss here only the supercharges which correspond to four dimensional supersymmetries, as opposed to superconformal transformations).

4.4 The linearized equations, zero mode and obstruction

Our method to find nearby solutions is to expand the fields and Killing spinor as a series in a perturbation parameter h and to solve them order by order. To begin, we need to discuss the small (linearized) fluctuations around the $AdS_5 \times S^5$ solution. The spectrum of these was found by Kim, Romans and van-Nieuwenhuizen [14]. Since we want to preserve the isometries of AdS_5, our solution will only involve turning on AdS scalars, whose spectrum is summarized in figure 1.

Since we require that only the minimal amount of supersymmetry be preserved (4d $\mathcal{N} = 1$ superconformal) we choose one specific spinor $\chi^{(0)} = \chi_0$ out of the 4 (positive chirality) χ's, to be the zeroth order spinor corresponding to the preserved supercharges. This choice breaks the global symmetry $\mathrm{SO}(6)_R \to \mathrm{U}(1)_R \times \mathrm{SU}(3)$, with $\mathbf{4} \to \mathbf{3}_1 + \mathbf{1}_{-3}$. It is convenient to use complex coordinates in $\mathbb{R}^6 \simeq \mathbb{C}^3$, replacing the SO(6) indices $\alpha, \beta, \ldots$, by SU(3) indices $i, j, \bar{\imath}, \bar{\jmath}$. We choose the flat metric to be of the form $ds_0^2 = 2\,dz^i\,d\bar{z}^{\bar{\imath}}$, namely we normalize the complex coordinates such that $\eta_{i\bar{\imath}} = \eta_{\bar{\imath}i} = 1$ and other components vanish.

[17] These equations are the same as in [12] since $\Gamma^4\chi = \Gamma^7\chi$.

This normalization allows us to do away with all barred indices — they can always be lowered or raised by η and become unbarred with no additional factors (the price being the factor of 2 in the expression for ds_0^2).

Having chosen a complex structure, we can identify the gamma matrices with creation and annihilation operators. We will use lower case gamma matrices $\gamma^\dagger{}_j, \gamma^{\dagger i}$ to denote the gamma matrices in the complex coordinates, satisfying $\{\gamma^i, \gamma^\dagger{}_j\} = 2\,\delta^i_j$, and we will choose our coordinates such that χ_0 is identified with the Clifford vacuum, $\gamma^\dagger{}_i\,\chi_0 = 0$ (and by complex conjugation, $\gamma^j \chi_0^* = 0$). The dagger on $\gamma_j^\dagger$ with lower indices is a redundant notation to remind us that $\gamma^\dagger{}_j$ is a creation operator with respect to the "state" χ_0^*. Using the complex coordinates we can easily formulate the requirement for $\mathrm{U}(1)_R$ isometry by requiring that each term in the expansion of the fields has an equal number of z's and $\bar{z}$'s.[18]

Next, we will linearize our equations (4.30) around the $AdS_5 \times S^5$ solution. This will give us the equations for the leading deformation around the original solution, and at every order n in the perturbation series we will get the same form of equations for the n'th order fields, with additional contributions coming from the lower order fields. At the linearized level the metric is flat, and so $\Gamma_\alpha \simeq \Gamma_{\hat{\alpha}}$. The equations (4.30), and therefore also the linearized equations, may be split into components whose 4d spinor is proportional to ζ or to ζ^*. This decomposition turns out to distinguish odd and even orders since the perturbation parameters h (which correspond to a particular mode of the 3-form field G) are odd under the element $(-\mathbf{1}) \in \mathrm{SL}(2,\mathbb{Z})$ which inverts G and keeps the other bosonic fields invariant, and the original $AdS_5 \times S^5$ solution is invariant under this $\mathbb{Z}_2$. The $\mathbb{Z}_2$ is actually a $\mathbb{Z}_4$ when it acts on the fermionic fields, multiplying them (and the spinor χ) by $\pm i$ (depending on their Γ_{11} chirality); this has the same effect as taking $\chi \to \chi$, $\chi^* \to -\chi^*$. This implies that at odd orders the only bosonic field that can be turned on is G, and the only fermionic modes that will be turned on are $\chi^{(o)}$ obeying $\Gamma^7\,\chi^{(o)} = -\chi(o)$, while at even orders g,B, and ρ get contributions, as well as $\chi^{(e)}$ which obeys $\Gamma^7\,\chi^{(e)} = +\chi^{(e)}$. The linearized equations at odd orders are thus of the form:

$$0 = \delta\psi_\alpha = \left(\partial_\alpha + \frac{\Gamma_r\,\Gamma_\alpha}{r}\right)\chi^{(o)} - \frac{1}{96\,r}(2\,\not{G}\,\Gamma_\alpha + \Gamma_\alpha\,\not{G})\chi_0^* + \cdots, \tag{4.32}$$

$$0 = \delta\psi = \frac{1}{r}\left(\Gamma_r\,\chi^{(o)} + \frac{1}{96}\,\not{G}\,\chi_0^*\right) + \cdots, \tag{4.33}$$

$$0 = \delta\lambda = -\frac{4}{96}\,\not{G}\,\chi_0 + \cdots, \tag{4.34}$$

where the $\cdots$ refers to contributions involving fields of lower order in the deformation parameter.

Similarly, the linearized equations at even orders are:

$$0 = \delta\psi_\alpha = \partial_\alpha\chi^{(e)} + \frac{1}{4}\,\not{\psi}_\alpha\,\chi_0 + \frac{1}{2}\left[-\not{\partial}\rho + \frac{(4\rho - \delta f_5)}{r}\Gamma_r\right]\Gamma_\alpha\chi_0 + \cdots\,, \tag{4.35}$$

[18]Through the Hopf fibration we can think of S^5 as a circle fibration over $\mathbf{CP}^2$. The fact that the $\mathrm{U}(1)_R$ isometry stays intact throughout the deformation means that the problem can be described as a deformation of $\mathbf{CP}^2$ rather than S^5, but we will not use this description here.

$$0 = \delta\psi = \frac{1}{2}\left[r\partial_\alpha\rho\,\Gamma^{\hat{\alpha}} + (4\rho - \delta f_5)\,\Gamma_r\right]\chi_0 + \cdots\,, \tag{4.36}$$

$$0 = \delta\lambda = \partial\!\!\!/ B\,\chi_0^* + \cdots\,, \tag{4.37}$$

where we defined $f_5 = 1 + \delta f_5$, and we also need to impose the flux quantization constraint

$$0 = \delta f_5 + \int_{S^5} \mathrm{tr}(g_{\alpha\beta} - \eta_{\alpha\beta})\,d^5x + \cdots\,. \tag{4.38}$$

For later use we linearize $\delta\tilde{\psi}$ as well. At odd orders

$$\delta\tilde{\psi} = \left(\partial\!\!\!/ + \frac{2}{r}\Gamma_r\right)\chi + \cdots \tag{4.39}$$

While at even orders

$$\delta\tilde{\psi} = \partial\!\!\!/\chi + \left[\phi\!\!\!/_6 - \frac{i}{2}Q\!\!\!/ + \left(\frac{1}{r}\right)\Gamma_r\,(4\,\rho - f_5) + 5\,\partial\!\!\!/\rho\right]\chi_0\cdots \tag{4.40}$$

where $\phi\!\!\!/_6 = \phi_\alpha\,\Gamma^\alpha$.

The odd equations have a zero mode that will turn out to be crucial for our analysis, and we will describe it in detail in subsection 4.7.

4.5 First order

Now, we have everything we need in order to perform an order by order analysis of the equations. A field ϕ at order (n) will be denoted by $\phi^{(n)}$.

The first order analysis was performed already in [12]. From the spectrum of scalars we know that the only massless ones are the $\mathbf{1}_{\mathbb{C}}$ from B, the $\mathbf{45}_{\mathbb{C}}$ from G and the $\mathbf{105}_{\mathbb{R}}$ from the metric and 5-form. Out of these marginal modes the only one (except for B) which is super-marginal (namely, supersymmetric and marginal) is in the $\mathbf{10}_0$ representation, which appears in the $\mathbf{45}$ when we decompose it under $\mathrm{SO}(6) \to \mathrm{SU}(3)\times\mathrm{U}(1)_R$ (see appendix B for our group theory conventions). This mode comes from G so it is odd under the $\mathbb{Z}_2$ discussed above. To parameterize the coefficient of this mode we introduce the perturbation parameters h_{ijk}, which are a 3rd rank symmetric tensor of SU(3). h will be proportional to the superpotential deformation in the field theory dual, and we will determine the precise relation between them in the next section.

At this point we specialize our notation to the perturbation problem. As mentioned already we work in the untilded frame where the metric is flat at zeroth order $\eta_{i\bar{j}} = \delta_{ij}$, so $r^2 = 2\,z^i\,\bar{z}_i$. All indices are raised and lowered by means of $\eta_{i\bar{j}}$, and all gamma matrices are expressed by lower case γ's which are defined in an orthonormal frame. We define h tensors with less than 3 indices by contracting with $z's$, namely $h_{ij} \equiv h_{ijk}\,z^k$, $h_i \equiv h_{ijk}\,z^j\,z^k$, $h \equiv h_{ijk}\,z^i\,z^j\,z^k$,$\bar{h}^{ij} \equiv \bar{h}^{ijk}\,\bar{z}_k\,,\ldots$ (where $\bar{h}^{ijk}$ is the complex conjugate of h_{ijk}). Indices are antisymmetrized without any prefactor, $h_{[i}\bar{z}_{l]} = h_i\,\bar{z}_l - h_l\,\bar{z}_i$. We record here a useful relation $\gamma^\dagger{}_{123}\,\gamma^{123}\,\chi_0 = -8\;\chi_0$. We also define

$$\gamma_r = \frac{(\bar{z}_i\,\gamma^i + z^i\,\gamma^\dagger{}_i)}{r}\,. \tag{4.41}$$

The explicit form of $G^{(1)}$ in terms of h, corresponding to turning on the massless scalar in the $\mathbf{10_0}$ representation, is quite involved. It can be deduced from the expressions for a general superpotential in [12] to be

$$\frac{1}{96}\,G^{(1)} = \frac{1}{6}h\,\epsilon^{ijk}\,\frac{d\bar{z}_{ijk}}{r^6} - \left(\frac{1}{2}r^2\,h_{il} + h_{[i}\,\bar{z}_{l]}\right)\frac{1}{2}\epsilon^{jkl}\frac{d\bar{z}_{jk}\,dz^i}{r^6} + \\ +h_{il}\,\bar{z}_j\,\bar{z}_m\,\epsilon^{lmk}\,\frac{dz^{ij}\,d\bar{z}_k}{r^6}\,, \tag{4.42}$$

where a factor of 1/96 was inserted in the normalization of h for later convenience. This expression can be derived by noting that from equation (4.34) we get

$$0 = \not{G}\,\chi_0 = 6\,G_{123}\,\gamma^{123}\,\chi_0 + G^i{}_{jk}\left(2\,\gamma_i\,\gamma^{jk} + \gamma^{[k}\,\gamma_i\,\gamma^{j]} + 2\,\gamma^{jk}\,\gamma_i\right)\chi_0\,, \tag{4.43}$$

and since both terms have to vanish we get $G_{123} = 0$, $G^j{}_{ij} = 0$. Similarly one continues to analyze the other equations for G to arrive at (4.42).

Slashing $G^{(1)}$ one gets

$$\boxed{\begin{aligned}\frac{1}{96}r^3\,\not{G}^{(1)} = {}& \frac{1}{6}\,h\,\epsilon^{ijk}\,\gamma^\dagger{}_{ijk} - \left(\frac{1}{2}r^2\,h_{il} + h_{[i}\,\bar{z}_{l]}\right)\frac{1}{2}\epsilon^{jkl}\,\gamma^\dagger{}_{jk}\,\gamma^i + \\ &+2\,h_i\,\bar{z}_j\,\gamma^\dagger{}_k\,\epsilon^{ijk} + h_{il}\,\bar{z}_j\,\bar{z}_m\,\epsilon^{klm}\,\gamma^\dagger{}_k\,\gamma^{ij}\,.\end{aligned}} \tag{4.44}$$

The expression for $\chi^{(1)}$ can be found from equation (4.33),

$$\boxed{\chi^{(1)} = -\frac{1}{96}\,\gamma_r\,\not{G}^{(1)}\,\chi_0^* = -\frac{1}{2}\,h_i\,\gamma^i\,\gamma^\dagger{}_{123}\,\chi_0^*/r^2\,.} \tag{4.45}$$

Note that $\Gamma^7\,\chi^{(1)} = -\chi^{(1)}$ and that equation (4.32) is also automatically obeyed.

4.6 Second order

We now continue to expand the equations into second order. Here we found it convenient to perform our computations both by hand and by computerized symbolic computations using *Mathematica* [27]. The computer computation required implementing the Clifford algebra γ_i as well as the other tensors which appear here $h_{ijk}, \epsilon_{ijk}, \delta^i_j$ (see [28] for a related *Mathematica* package).

We start with $B^{(2)}$. From the third equation in (4.30) (whose linearized form is (4.37)) we have at second order $\not{\partial}B^{(2)}\,\chi_0^* = \frac{4}{96}(\not{G}\,\chi)^{(2)} = \frac{4}{96}\not{G}^{(1)}\,\chi^{(1)}$. We compute

$$\frac{1}{96}\not{G}^{(1)}\,\chi^{(1)} = -\frac{1}{96^2}\not{G}^{(1)}\,\gamma_r\,\not{G}^{(1)}\,\chi_0^* = -4\,h_{ij}h_k\,\bar{z}_m\bar{z}_n\,\epsilon^{ikm}\,\epsilon^{jln}\frac{\gamma^\dagger{}_l\,\chi_0^*}{r^5}\,. \tag{4.46}$$

The resulting differential equation is integrable and we get

$$\boxed{B^{(2)} = -2\,h_{ij}h_{kl}\bar{z}_m\bar{z}_n\frac{\epsilon^{ikm}\epsilon^{jln}}{r^4}\,.} \tag{4.47}$$

This expression is in the **27** representation of SU(3), which includes the modes with no $U(1)_R$ charge coming from the spherical harmonics of B in the **105** representation of SO(6). This can be seen from the fact that as a quadratic symmetric form in h it could apriori be in the $(\mathbf{10}\times\mathbf{10})_s = \mathbf{28}+\mathbf{27} = [6,0]+[2,2]$ representation, but the $\mathbf{28}=[6,0]$ is the symmetrization on all 6 indices which vanishes for our expression (due to the ϵ's), so it must be proportional to the **27**.

Turning next to ρ, we note that it appears in the linearized equations only in the combination $\rho' \equiv \rho - \delta f_5/4$. Moreover, equations (4.36), (4.30) give us an equation only for ρ' (recall that f_5 is a constant):

$$\frac{1}{2}(4\rho'\,\gamma_r + r\,\partial\!\!\!/\rho') = -\frac{1}{96}\,G\!\!\!/^{(1)}\,\chi^{(1)*} = \frac{1}{96^2}\,G\!\!\!/^{(1)}\,\gamma_r\,G\!\!\!/^{(1)*}\,\chi_0^*\,. \tag{4.48}$$

Using our first order results this is given by

$$\frac{1}{96^2}G\!\!\!/^{*(1)}\,\gamma_r\,G\!\!\!/^{(1)}\,\chi_0^* = 2\,h_i\bar{h}^{il}\,{\gamma^\dagger}_l\,\chi_0^*\,. \tag{4.49}$$

The resulting first order partial differential equation for $\rho'^{(2)}$,

$$4\,\rho'^{(2)}\frac{\bar{z}_m}{r^2} + \partial_m\rho'^{(2)} = 4\,h_{mj}\bar{h}^j/r^4\,, \tag{4.50}$$

is integrable and yields

$$\boxed{\rho'^{(2)} = \rho^{(2)} - \frac{1}{4}\delta f_5^{(2)} = 2\frac{h_i\bar{h}^i}{r^4}\,.} \tag{4.51}$$

Note that this expression is proportional to $V = |\partial W|^2$ for $W \propto h_{ijk}\,z^i z^j z^k$, which is the second order contribution to the field theory potential due to our deformation of the superpotential, and that the mode of ρ' in the **105** spherical harmonic is precisely the massless field which is identified with the fourth order scalar potential deformation in the field theory. Thus, supergravity automatically turns on the second order deformations in the field theory which are implied by the first order deformation and by supersymmetry.

The equation for the perturbations in the metric and the Killing spinor can now be deduced from equation (4.35),

$$\begin{aligned}\partial_\epsilon\chi^{(2)} + \partial_{[\alpha}S^{(2)}_{\beta]\epsilon}\,\gamma^{\alpha\beta}\,\chi_0 &= B_\epsilon\\ &= \frac{2}{96}G\!\!\!/^{(1)}\,\gamma_\epsilon\,\chi^{(1)*} - \left(\partial_\epsilon\rho'^{(2)} + 4\,\rho'^{(2)}\frac{z_\epsilon}{r^2}\right)\chi_0 + \partial\!\!\!/\rho\,\gamma_\epsilon\,\chi_0\,,\end{aligned} \tag{4.52}$$

where we defined S, a 2nd rank symmetric and real tensor, to be the correction to the frame and metric,

$$\begin{aligned}\Omega^{\hat\alpha}{}_\beta &= \delta^{\hat\alpha}{}_\beta - 4\,S_{\alpha\beta}\,,\\ &\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} - 8S_{\alpha\beta}\,,}\end{aligned} \tag{4.53}$$

and we defined

$$\begin{aligned}B_m &= -16\,h_{[m}\,\bar{z}_{p]}\,\bar{h}^p\frac{\chi_0}{r^6} + 8\,h_{i[m}\,\bar{z}_{p]}\,\bar{z}_j\,\bar{h}^p\,\gamma^{ij}\frac{\chi_0}{r^6} - 4\,h_{jm}\,\bar{h}^j\frac{\chi_0}{r^4}\,,\\ B^m &= 16\,\bar{h}^m\,h\frac{\chi_0}{r^6} - 8\,h_i\,\bar{z}_j\,\bar{h}^m\,\gamma^{ij}\frac{\chi_0}{r^6} + 2\,\partial_j\left(h_i\frac{\bar{h}^i}{r^4}\right)\gamma^{jm}\,\chi_0 + 4\,h_j\,\bar{h}^{jm}\frac{\chi_0}{r^4} -\\ &\quad -16\,h_i\,\bar{h}^i\,z^m\frac{\chi_0}{r^6}\,.\end{aligned} \tag{4.54}$$

We find that the $\gamma^{ij}\,\chi_0$ terms in (4.52) can be matched by

$$\boxed{\begin{aligned} S^{(2)}_{ij} &= \frac{h_{ij}\,\bar{h}}{r^4} - 2\,\bar{z}_i\bar{z}_j\,\frac{h_p\bar{h}^p}{r^6} + \partial_i\partial_j K\,, \\ S^{ij\,(2)} &= \overline{S^{(2)}_{ij}}\,, \\ S^{i}{}_{j}{}^{(2)} &= -\frac{\bar{h}^i\,h_j}{r^4} + \delta^i{}_j\,\frac{h_k\,\bar{h}^k}{r^4} + \frac{1}{8}\left(h_{jkl}\,\bar{h}^{ikl} - \frac{1}{3}\delta^i_j\,h_{klm}\bar{h}^{klm}\right), \end{aligned}} \tag{4.55}$$

where K is to be determined, and without any contribution involving $\chi^{(2)}$. Our supersymmetry equations depend only on derivatives of S, so we can always add arbitrary constant terms to S, as we did in our expression for S^i_j; the reason for adding the particular constant terms that we added will be explained below. We record for later use that

$$S^{j}_{j}{}^{(2)} = 2\frac{\bar{h}^j h_j}{r^4}\,. \tag{4.56}$$

The χ_0 term in (4.52) now implies

$$\begin{aligned} \partial_m\left(\hat{\chi}^{(2)} + 2\,\partial^j\partial_j K\right) &= \partial_m\left(-5\frac{h_i\bar{h}^i}{r^4}\right), \\ \partial^m\left(\hat{\chi}^{(2)} - 2\partial^j\partial_j K\right) &= \partial^m\left(5\frac{h_i\,\bar{h}^i}{r^4}\right), \end{aligned} \tag{4.57}$$

where we defined $\chi^{(2)} = \hat{\chi}^{(2)}\,\chi_0$. This leads to

$$2\,\partial^j\partial_j K = \Delta K = -5\frac{h_j\bar{h}^j}{r^4}\,, \tag{4.58}$$

$$\boxed{\chi^{(2)} = 0.} \tag{4.59}$$

The fact that $\chi^{(2)}$ vanishes is somewhat surprising, and could be a hint about a general feature of the full solution.

Before solving explicitly (4.58) for K let us describe in some detail the general solution to a Poisson equation, $\Delta K = \psi$, where ψ is some source function independent of the radial coordinate r. Decomposing ψ and K into spherical harmonics[19] Y_{lm} on S^5, by writing $K = \sum_{lm} K_{lm}(r)Y_{lm}$, $\psi = \sum_{lm}\psi_{lm}Y_{lm}$, we find

$$\sum Y_{lm}\left[\partial_r^2 + \frac{5}{r}\partial_r - \frac{1}{r^2}\,l\,(l+4)\right]K_{lm} = \Delta K = \psi = \sum \psi_{lm}\,Y_{lm}\,. \tag{4.60}$$

The solution is

$$K_{lm} = \begin{cases} \psi_{lm}\dfrac{r^2}{(12 - l(l+4))} & l \neq 2\,, \\ \psi_{lm}\dfrac{r^2\log(r/r_0)}{8} & l = 2\,. \end{cases} \tag{4.61}$$

[19]The index m runs over all the states within the representation l.

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Note that the $l=2$ mode stands out as it generates logarithms in K. The explicit solution of equation (4.58) for K is

$$\boxed{K = \frac{1}{4}\frac{h_i \bar{h}^i}{r^2} - \frac{1}{48}(6\, h_{ij}\bar{h}^{ij} - h_{ijk}\bar{h}^{ijk}\, r^2)\log\left(\frac{r^2}{r_0^2}\right) - \frac{1}{6}\, h_{ij}\bar{h}^{ij},} \tag{4.62}$$

for arbitrary r_0 (the $log(r_0)$ term multiplies a zero mode of the laplacian).

Let us check whether this is consistent with the conformal isometries. The first condition in (4.8) is satisfied, as it is easy to check that the logarithm does not appear in $\partial_i\partial_j K$ which appears in the metric. We require also that the metric satisfies the second constraint of (4.8). The linearized constraint is

$$S_{i\alpha}\, x^\alpha = 0 \tag{4.63}$$

(a similar equation with upper index i is the complex conjugate of this one). We find

$$\begin{aligned}
S_{i\alpha}\, x^\alpha &= S_{ij}\, z^j + S_i{}^j\, \bar{z}_j \\
&= \frac{h_i \bar{h}}{r^4} - \bar{z}_i \frac{h_p \bar{h}^p}{r^4} + z^j\, \partial_i\partial_j K - \\
&\quad - \frac{h_i \bar{h}}{r^4} + \bar{z}_i \frac{h_p \bar{h}^p}{r^4} + \frac{1}{8}\left(h_{ikl}\bar{h}^{jkl} - \frac{1}{3}\delta_i^j\, h_{klm}\bar{h}^{klm}\right)\bar{z}_j \\
&= z^j\, \partial_i\partial_j K + \frac{1}{8}\left(h_{ikl}\bar{h}^{kl} - \frac{1}{3}\bar{z}_i h_{klm}\bar{h}^{klm}\right), \\
z^j\, \partial_i\partial_j K &= -z^j\, \partial_i\partial_j \left[\frac{1}{48}\,(6\, h_{kl}\bar{h}^{kl} - h_{klm}\bar{h}^{klm}\, r^2)\log\left(\frac{r^2}{r_0^2}\right)\right] \\
&= -\frac{1}{48}\left(6\, h_{ikl}\bar{h}^{kl} - 2\, h_{klm}\bar{h}^{klm}\, \bar{z}_i\right).
\end{aligned} \tag{4.64}$$

We see that exactly for the choice we made of the constant terms in S^i_j, this equation vanishes identically and our solution is conformally invariant and supersymmetric. This agrees with our field theory discussion in section 2, where we argued that if one swallows the wave function renormalization into the normalization of the chiral superfields, we expect to find a conformal solution to second order in the deformation whether or not $(h\bar{h})_{\mathbf{8}}$ vanishes.

It is interesting to note that the constant terms we added to S^i_j are exactly proportional to $(h\bar{h})_{\mathbf{8}}$. These terms seem to be related to a diffeomorphism of the form $z^i \to z^i + \frac{1}{16}[(h\bar{h})_{\mathbf{8}}]^i_j z^j$; if instead we would perform the same diffeomorphism with an imaginary coefficient, this would correspond (from the AdS_5 point of view) to an SU(3) gauge transformation with a gauge parameter $(h\bar{h})_{\mathbf{8}}\log(r/r_0)$. This suggests that these terms are related to a complexified SU(3) transformation, and we saw at the end of section 2 that in the field theory these transformations are precisely related to the wave function renormalization induced by the anomalous dimensions. Thus, at this order we seem to find a nice agreement between the field theory and supergravity results, assuming that supergravity corresponds to the field theory where we rescaled the fields to have canonical kinetic terms. We will make this more precise at third order, when we will compute the running of the coupling constant (which is a physically measurable quantity that can be directly compared between field theory and supergravity).

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We can also solve for the perturbation in the normalization constant $\delta f_5^{(2)}$, although it will not be necessary for any other computation presented in this paper. The flux normalization equation at second order is

$$
\begin{aligned}
0 &= \delta f_5^{(2)} - \int_{S^5} 8\, S_j^{j\,(2)}\, d^5x \Rightarrow \\
\Rightarrow \delta f_5^{(2)} &= 16 \int_{S^5} \frac{h_i\, \bar{h}^i}{r^4}\, d^5x = \frac{2}{3}|h|^2 \,,
\end{aligned}
\tag{4.65}
$$

where we used the definition of S (4.53), $\operatorname{tr}(S) = 2\, S^j{}_j$ and the integration formula (4.79), and we defined $|h|^2 \equiv h_{ijk}\, \bar{h}^{ijk}$ which is a z independent constant.

4.7 $\delta\tilde{\psi}$ at third order

In equation (4.29) we introduced the linear combination of SUSY variation equations

$$
\delta\tilde{\psi} = \Gamma^\alpha\, \delta\psi_\alpha + 6\, \delta\psi = \left[\not{D} - \frac{i}{2}\not{Q} + \left(\frac{1}{r}\right) \Gamma_r\, (1 - f_5\, \exp(-4\,\rho)\, \Gamma^7) + 5\, \not{\partial}\rho \right] \chi = 0 \,, \tag{4.66}
$$

where $\not{D} = \not{\partial} + \not{\omega}_6$, $\not{\omega}_6 = \frac{1}{4}\, \omega_{\alpha,\hat{\beta}\hat{\gamma}}\, \Gamma^\alpha\, \gamma^{\hat{\beta}\hat{\gamma}}$.

We will be interested in solving this equation at third order in the deformation, where this term will be responsible for eliminating the marginal but not exactly-marginal deformations, but let us consider first the lower orders, including the even orders. At zeroth order we have

$$
0 = \delta\tilde{\psi}^{(0)} = \not{\partial}\chi_0 \,, \tag{4.67}
$$

which is clearly satisfied. At first order we find

$$
0 = \delta\tilde{\psi}^{(1)} = \left(\not{\partial} + \frac{2}{r}\gamma_r \right) \chi^{(1)} \,. \tag{4.68}
$$

This is an SO(6) invariant equation which requires $\chi^{(1)}$ to be a zero mode of the differential operator $(\not{\partial} + \frac{2}{r}\gamma_r)$. This operator has a single zero mode which is in the in the $\overline{\mathbf{60}} = [0,2,1]$ representation of SO(6). The $\overline{\mathbf{60}}$ includes two SU(3) irreducible representations with the correct R-charge (+3), which are in the **8** and **10** representations of SU(3). We can write them down explicitly and check that they solve (4.68):

$$
\begin{aligned}
(ZM\mathbf{8})^i_j &= \left(12 \frac{z^i \bar{z}_j}{r^2} - 2\, \delta^i_j \right) \chi_0^* - \epsilon_{mnj}\, z^i z^n\, \gamma^m \gamma^\dagger{}_{123} \frac{\chi_0^*}{r^2} \,, \\
(ZM\mathbf{10})^{ijk} &= z^{(i} z^j\, \gamma^{k)} \gamma^\dagger{}_{123} \frac{\chi_0^*}{r^2} \,.
\end{aligned}
\tag{4.69}
$$

In our solution above, $\chi^{(1)}$ (4.45) was precisely proportional to $h_{ijk}(ZM\mathbf{10})^{ijk}$, as required by (4.68). Let us also record these quantities after being multiplied by γ_r:

$$
\begin{aligned}
\gamma_r (ZM\mathbf{8})^i_j &= \left(16\, z^i \bar{z}_j z^k - 2\, z^i \delta^k_j - 2\, z^k \delta^i_j \right) \gamma^\dagger{}_k\, \chi_0^* \,, \\
\gamma_r (ZM\mathbf{10})^{ijk} &= \frac{2}{r^3} \left(z^{(i} z^j z^{k)}\, \gamma^\dagger{}_{123}\, \chi_0^* + 2\, \epsilon^{pq(i}\, z^j z^{k)} \bar{z}_p\, \gamma^\dagger{}_q\, \chi_0^* \right) .
\end{aligned}
\tag{4.70}
$$

At second order we find

$$0 = \delta\tilde{\psi}^{(2)} = \partial\!\!\!/\chi^{(2)} + \left[\tilde{\partial\!\!\!/}_6 + 5\,\partial\!\!\!/\rho + 4\,\rho'\frac{\gamma_r}{r}\right]^{(2)} \chi_0 \,. \tag{4.71}$$

This expression is consistent with our previous results $\tilde{\partial\!\!\!/}_6 = -4\rho'\gamma_r/r - 5\,\partial\!\!\!/\rho^{(2)}$, which follows from the expression for $S^{(2)}$, and $\chi^{(2)} = 0$.

At third order we have

$$0 = \delta\tilde{\psi}^{(3)} = \left(\partial\!\!\!/ + \frac{2}{r}\gamma_r\right)\chi^{(3)} + \left[\tilde{\partial\!\!\!/}_6 + 5\,\partial\!\!\!/\rho - 4\rho'\frac{\gamma_r}{r}\right]^{(2)} \chi^{(1)} + St\,, \tag{4.72}$$

where the term St is defined by

$$St = 4\,S^{\alpha\beta}\,\gamma_\beta\left(\partial_\alpha + 2\frac{\partial_\alpha r}{r}\right)\chi^{(1)}\,. \tag{4.73}$$

An explicit calculation gives

$$\begin{aligned} St = &\left(16\,h_i\,h_j\,\bar{h}^{ij} - 12\,h_{ij}\,\bar{h}^{ij}\,h - \frac{2}{3}\,h\,h_{ijk}\,\bar{h}^{ijk}\right)\gamma^\dagger{}_{123}\frac{\chi_0^*}{r^6} + \\ &+ \left(-16\,h_i\,h_{jk}\,\bar{h}^k + 2r^4\bar{h}^{kmn}h_{imn}h_{jk}\right)\epsilon^{ijl}\,\gamma^\dagger{}_l\frac{\chi_0^*}{r^6}\,. \end{aligned} \tag{4.74}$$

As we saw above, the differential operator $(\partial\!\!\!/ + \frac{2}{r}\gamma_r)$ has a zero mode and its image is not the full space. We will denote

$$\delta\tilde{\psi}^{(3)} = \left(\partial\!\!\!/ + \frac{2}{r}\gamma_r\right)\chi^{(3)} + Obst\,, \tag{4.75}$$

and if $Obst$ will have a component which is not in the image of the differential operator $(\partial\!\!\!/ + \frac{2}{r}\gamma_r)$ (acting on spinor spherical harmonics on S^5), the equation will have no AdS-invariant solution. Since we can choose a basis of spherical harmonics which are eigenfunctions of $(\partial\!\!\!/ + \frac{2}{r}\gamma_r)$ up to multiplication by γ_r (namely, they obey $(\partial\!\!\!/ + \frac{2}{r}\gamma_r)\chi = \alpha\gamma_r\chi/r$), the requirement is that $Obst$ will have no components proportional to the spherical harmonics appearing in (4.70).

Substituting the expressions for St and $\rho'^{(2)}$ into (4.72) we can evaluate $Obst$ and $\gamma_r Obst$ which will be useful shortly:

$$\begin{aligned} r\,Obst = &\left(16\,\frac{h_ih_j\bar{h}^{ij}}{r^5} + 16\,\frac{hh_k\bar{h}^k}{r^7} - 12\,\frac{hh_{ij}\bar{h}^{ij}}{r^5} - \frac{2}{3}\,\frac{h|h|^2}{r^3}\right)\gamma^\dagger{}_{123}\chi_0^* + \\ &+ \left(16\,\frac{h_{ik}\bar{h}^kh_j}{r^5} - 32\,\frac{\bar{z}_i\,h_jh_k\bar{h}^k}{r^7} + 2\frac{\bar{h}^{kmn}h_{imn}h_{jk}}{r}\right)\epsilon^{ijl}\,\gamma^\dagger{}_l\chi_0^*\,, \end{aligned} \tag{4.76}$$

$$\begin{aligned} r\,\gamma_r\,Obst = &\left(-32\,h_ih_{jk}\bar{h}^k + 4r^4\bar{h}^{kmn}h_{imn}h_{jk}\right)\bar{z}_l\,\epsilon^{ijl}\frac{\chi_0^*}{r^6} + \\ &+ \Big[16h_jh_k\bar{h}^{jk}\bar{z}_i + 8h\,h_{ij}\bar{h}^j - 12hh_{jk}\bar{h}^{jk}\,\bar{z}_i - \frac{2}{3}r^2\,h\,|h|^2\bar{z}_i + \\ &+ r^4\bar{h}^{klm}h_{ilm}h_k - r^4\bar{h}^{klm}h_{lm}h_{ik}\Big]\gamma^i\gamma^\dagger{}_{123}\frac{\chi_0^*}{r^6}\,. \end{aligned} \tag{4.77}$$

4.8 Projections of *Obst* and the obstruction to superconformality

In order to check whether we can solve (4.72) with a conformally invariant solution we need to project the spinor *Obst* onto the modes (4.70), and check whether this projection vanishes or not. The equations for the vanishing of this projection give two cubic equations for h (actually $h\,h\,\bar{h}$), one in the **10** representation and the other in the **8**. In order to match with the field theory both of these two equations must have $(h\bar{h})_{\mathbf{8}}$ as a factor. For the cubic **8** this turns out to be trivial since there is a unique cubic expression in this representation, which factorizes (as shown explicitly below). For the **10** the situation is more interesting: two different expressions can be formed out of h and only a single linear combination factorizes. As we will see in this section, it is exactly this combination that happens to appear in the projection of *Obst*. This is a non-trivial test of the AdS/CFT correspondence.

In general the projection of a spinor χ onto a mode χ_i is given by the inner product

$$(\chi_i, \chi) = \int_{S^5} \bar{\chi}_i\, \chi\,. \tag{4.78}$$

A useful formula is

$$\int_{S^5} z^{i_1} \cdots z^{i_n}\, \bar{z}_{j_1} \cdots \bar{z}_{j_n} = \sum_{\sigma \in S_n} \frac{\delta_{j_1}^{i_{\sigma(1)}} \cdots \delta_{j_n}^{i_{\sigma(n)}}}{(2^{n-1}(n+2)!)}\,, \tag{4.79}$$

where S_n is the permutation group on n objects and we have normalized $\int_{S^5} 1 = 1$. The index structure is determined by SU(3) invariance, and the normalization factor can be determined by multiplying both sides by $2^n\,\delta_{j_1}^{i_1} \cdots \delta_{j_n}^{i_n}$.[20]

Let us first compute the projection onto the **10**. The result must be a combination of the **10**s which can be made out of $h\,h\,\bar{h}$. Actually there are two such combinations

$$\begin{aligned} \mathbf{10}' &= h_{pq(i} h_{jk)r} \bar{h}^{pqr}\,, \\ \mathbf{10}'' &= h_{(ijk)} h_{pqr} \bar{h}^{pqr}\,, \end{aligned} \tag{4.80}$$

where the brackets () denote symmetrization. We can form a linear combination of the two from the product of $(h\bar{h})_{\mathbf{8}} = \bar{h}^{plm}\,h_{klm} - \frac{1}{3}\,\delta_k^p \bar{h}^{lmn}\,h_{lmn}$ with h:

$$\mathbf{10}''' = \left[\left(\bar{h}^{plm}\,h_{klm} - \frac{1}{3}\,\delta_k^p\,\bar{h}^{lmn}\,h_{lmn}\right) h_{pij} + (ijk \text{ perm})\right] = \mathbf{10}' - \frac{1}{3}\mathbf{10}''\,. \tag{4.81}$$

We can express the different terms appearing in the projection $(ZM\mathbf{10}, \gamma_r\, Obst)$ as linear combinations of $\mathbf{10}'$ and $\mathbf{10}''$:

$$\left(ZM\mathbf{10}, h_j h_k\, \bar{h}^{jk}\, \bar{z}_i\, \gamma^i {\gamma^\dagger}_{123} \frac{\chi_0^*}{r^6}\right) = \frac{(24\ \mathbf{10}' + 0\ \mathbf{10}'')}{(8 \cdot 6!)},$$

[20] The formula, generalized to an arbitrary complex dimension d, $1 \le i_s, j_s \le d$, $1 \le s \le n$ (here $d = 3$) is $\int_{S^{(2d-1)}} z^{i_1} \cdots z^{i_n}\, \bar{z}_{j_1} \cdots \bar{z}_{j_n} = \sum_{\sigma \in S_n} \delta_{j_1}^{i_{\sigma(1)}} \cdots \delta_{j_n}^{i_{\sigma(n)}} / (2^n c_{n,d})$. The normalization constant is given by $c_{n,d} = \sum_{(i_1,\ldots,i_n)}^{d^n} \sum_{\sigma \in S_n}^{n!} \delta_{i_1}^{i_{\sigma(1)}} \cdot \cdots \cdot \delta_{i_n}^{i_{\sigma(n)}} = \sum_{i*}^{d^n} n_1! \cdot \cdots \cdot n_d! = \sum_{p*} n! = \frac{(n+d-1)!}{n!(d-1)!} n! = \frac{(n+d-1)!}{(d-1)!}$, where $i*$ is any set of indices $(i_1, \ldots, i_n)$ such that there are n_1 1's and $\ldots n_d$ d's, and $p*$ is any partition of n objects (the indices) into d bins (the possible values).

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$$\left(ZM\mathbf{10}, h\, h_{ij}\, \bar{h}^j\, \gamma^i \gamma^\dagger{}_{123}\, \frac{\chi_0^*}{r^6} \right) = \frac{(24\ \mathbf{10}' + 0\ \mathbf{10}'')}{(8 \cdot 6!)},$$
$$\left(ZM\mathbf{10}, h\, h_{jk}\, \bar{h}^{jk}\, \bar{z}_i\, \gamma^i \gamma^\dagger{}_{123}\, \frac{\chi_0^*}{r^6} \right) = \frac{(18\ \mathbf{10}' + 6\ \mathbf{10}'')}{(8 \cdot 6!)},$$
$$\left(ZM\mathbf{10}, h\, h_{klm}\, \bar{h}^{klm}\, \bar{z}_i\, \gamma^i \gamma^\dagger{}_{123}\, \frac{\chi_0^*}{r^4} \right) = \frac{(0\ \mathbf{10}' + 6\ \mathbf{10}'')12}{(8 \cdot 6!)},$$
$$\left(ZM\mathbf{10}, h_k h_{ilm}\, \bar{h}^{klm}\, \gamma^i \gamma^\dagger{}_{123}\, \frac{\chi_0^*}{r^2} \right) = \frac{(2\ \mathbf{10}' + 0\ \mathbf{10}'')60}{(8 \cdot 6!)},$$
$$\left(ZM\mathbf{10}, h_{ik} h_{lm}\, \bar{h}^{klm}\, \gamma^i \gamma^\dagger{}_{123}\, \frac{\chi_0^*}{r^2} \right) = \frac{(2\ \mathbf{10}' + 0\ \mathbf{10}'')60}{(8 \cdot 6!)}. \tag{4.82}$$

After summing up all the contributions we find

$$\begin{aligned}(ZM\mathbf{10}, \gamma_r Obst) = \frac{1}{8 \cdot 6!}\Big[& 16(24\,\mathbf{10}' + 0\,\mathbf{10}'') + 8(24\,\mathbf{10}' + 0\,\mathbf{10}'') - 12(18\,\mathbf{10}' + 6\,\mathbf{10}'') - \\ & - \frac{2}{3}12(0\,\mathbf{10}' + 6\,\mathbf{10}'') + 60(2\,\mathbf{10}' + 0\,\mathbf{10}'') - 60(2\,\mathbf{10}' + 0\,\mathbf{10}'')\Big] \\ = \frac{12}{8 \cdot 6!} \cdot (30\mathbf{10}' - & 10\mathbf{10}'') = \frac{30 \cdot 12}{8 \cdot 6!}\mathbf{10}'''\,. \end{aligned} \tag{4.83}$$

As expected from field theory, the projection depends solely on $\mathbf{10}'''$! Thus, it vanishes if and only if $(h\bar{h})_\mathbf{8}$ vanishes. In the framework of our computation the fact that the ratio of the two terms is exactly (-3) is miraculous, though we expect (because of the correspondence with field theory) that it "had to happen" due to a deeper principle which is not manifest here. In order to write down the projection itself one needs to multiply by a normalization factor of 4!, and we find that the projection is given by

$$\gamma_r\, Obst \to \frac{3}{2} \left(\bar{h}^{plm} h_{klm} - \frac{1}{3}\, \delta_k^p\, \bar{h}^{lmn} h_{lmn} \right) h_{pij}\, z^i z^j\, \gamma^k \gamma^\dagger{}_{123} \frac{\chi_0^*}{r^3}\,. \tag{4.84}$$

Similarly, we compute the projection on $ZM\mathbf{8}$. Here we can form only one $\mathbf{8}$ out of $h\,h\,\bar{h}$ and it has $(h\bar{h})_\mathbf{8}$ as a factor,

$$\mathbf{8}^i_j = \epsilon^{ikl}\, \bar{h}^{pqr}\, h_{jkp}\, h_{lqr} = \epsilon^{ikl}\, [(h\bar{h})_\mathbf{8}]^m_k\, h_{jlm}\,. \tag{4.85}$$

There are various terms in $Obst$ that contribute to the projection, but we find that the different contributions cancel each other and we have

$$(\gamma_r ZM\mathbf{8}, Obst) = 0\,. \tag{4.86}$$

Thus, as expected, we find an obstruction to constructing a conformally invariant solution at third order if and only if $(h\bar{h})_\mathbf{8}$ is non-zero. We expect that when this vanishes we can construct a conformally invariant solution to all orders, but we have not been able to prove this explicitly.

5. Comparison to field theory and non-renormalization

In section 4 we found that when the combination $(h\bar{h})_\mathbf{8}$ is non-zero, we have an obstruction to solving the third order SUSY equations with a conformally invariant solution. This is exactly what we expect from our field theory analysis in section 2. In this section we will make the comparison more precise by comparing the logarithmic running of the coupling constants in the field theory (at one-loop) and in supergravity.

The obstruction term which we find at third order leads to a logarithmic term in $\chi^{(3)}$, which using (4.84), (4.75) is of the form

$$\chi^{(3)}_{log} = -\frac{3}{2}\log\left(\frac{r}{r_0}\right) h^{(3)}_{ijk} z^j z^k \gamma^i \gamma^\dagger_{123} \frac{\chi_0^*}{r^2}\,, \tag{5.1}$$

where we denoted the "bad" combination $\mathbf{10}'''$ by

$$6h^{(3)}_{ijk} = \left(\bar{h}^{plm} h_{klm} - \frac{1}{3}\delta^p_k |h|^2\right) h_{ijp} + (\text{permutations of } i,j,k)\,. \tag{5.2}$$

Using our equation for $\delta\psi$ (4.33) we find that such logarithmic terms in $\chi^{(3)}$ map directly to logarithmic terms in $G^{(3)}$, of the same form as the original terms we had in $G^{(1)}$ (except for the logarithmic dependence on r). Since these original terms corresponded to the coupling constants h_{ijk}, it is natural to interpret the logarithmic corrections as corresponding to a logarithmic running of the coupling constants h_{ijk}. By solving for these terms in G, we find that up to third order the couplings run as

$$h_{ijk}(r) = h_{ijk} + 3h^{(3)}_{ijk}\log\left(\frac{r}{r_0}\right) + \cdots\,, \tag{5.3}$$

for some arbitrary r_0. Interpreting r as the scale at which we measure the running coupling, this is similar to the field theory expressions which we get from the beta function (2.3) and the gamma function (2.4). Of course, the solution we find in which G behaves as in (5.3) cannot be valid for all values of r, since eventually the logarithm becomes large and higher order terms cannot be neglected; this is similar to what happens in field theory when the coupling runs. But, we can assume that there is some range of values of r where the running coupling is small and the approximation (5.3) is good; this is similar to what we do in field theory when we have a running coupling, and we impose UV and IR cutoffs.

To make a precise comparison we need to relate h to the coupling constant appearing in the field theory superpotential (namely, to determine its normalization). For this comparison it will be convenient to return the powers of κ and R_0 which we have ignored until now; returning these we find that (5.3) becomes

$$h_{ijk}(r) = h_{ijk} + 3\frac{\kappa^2}{R_0^8} h^{(3)}_{ijk}\log\left(\frac{r}{r_0}\right) + \cdots\,. \tag{5.4}$$

One possible way to determine the normalization of h is by carefully normalizing the operators $\mathrm{tr}(\Phi^i\Phi^j\Phi^k)$ on both sides and using the fact that their 2-point function is not renormalized. We will choose a different approach, which is to look at a D3-brane probe in the background we computed above, and to compare the scalar potential on this probe with the scalar potential we get in the field theory when we turn on a single non-zero eigenvalue for the matrices Φ^i (which is identified with the position of the D3-brane probe). Note that even though the D3-brane is not static, we can trust the naive computation of the potential on the brane since we are assuming that h is very small, and the motion of the D3-brane will be a higher order effect. Possible corrections to the field theory Kähler potential will also only contribute at higher orders in h.

The D3-brane action in the approximation of slowly varying background fields is of the form

$$S_{D3} = T_{D3} \int d^4x \sqrt{\det(G_{\mu\nu} + 2\pi\alpha'(B_{\mu\nu} - F_{\mu\nu}))} - \kappa T_{D3} \int d^4x A_4 \,, \tag{5.5}$$

where $T_{D3} = 1/(g_s(2\pi)^3(\alpha')^2)$ is the D3-brane tension, and pull-backs of the bulk fields to the D3-brane worldvolume are implied everywhere.

First, from the metric term we can read off the kinetic term for the transverse positions $z^i, \bar{z}_i$ of the D3-brane to zeroth order in h, in the coordinate system we used above. We find that the kinetic term is precisely given by $T_{D3}\partial_\mu z^i \partial^\mu \bar{z}_i$ (all the powers of r/R_0 cancel).

Next, we can compute the potential for the D3-brane, by expanding the terms in the action with no derivatives to second order in h. By plugging into (5.5) the expressions we computed above, we find that the scalar potential is given by

$$V_{D3} = T_{D3}\frac{r^4}{R_0^4}(e^{4\rho} - f_5) \simeq T_{D3}\frac{r^4}{R_0^4}(4\rho^{(2)} - \delta f_5^{(2)}) = T_{D3}\frac{8\kappa^2}{R_0^{12}} h_{ijk} z^j z^k \bar{h}^{ilm} \bar{z}_l \bar{z}_m \,. \tag{5.6}$$

To compare to the field theory we need to set the normalizations of the field theory lagrangian. We will take the relevant terms in the deformed SYM lagrangian to be of the form

$$\frac{1}{g_{\rm YM}^2} \int d^4x \,\mathrm{tr}(\partial_\mu \Phi^i \partial^\mu \Phi_i^*) + \left[\frac{N}{3(g_{\rm YM}^2 N)^{3/2}} \int d^4x d^2\theta H_{ijk} \,\mathrm{tr}(\Phi^i \Phi^j \Phi^k) + \text{c.c.}\right], \tag{5.7}$$

where we determined the normalization of the field theory coupling constant H such that it couples to the chiral primary operator as normalized in the natural way for the 't Hooft large-N limit [29] (in other words, to have a good large-N limit we need to keep H_{ijk} constant and not h_{ijk}; the relation between the two is determined below). By comparing the kinetic terms, using $4\pi g_s = g_{\rm YM}^2$, we see that we should identify $z^i \simeq \sqrt{2}\pi\alpha'\Phi^i$. Plugging this into (5.6) we find that it is given by

$$V_{D3} = \frac{16\pi^2\kappa^2(\alpha')^2}{R_0^{12} g_{\rm YM}^2} h_{ijk} \Phi^j \Phi^k \bar{h}^{ilm} \Phi_l^* \Phi_m^* \,. \tag{5.8}$$

Meanwhile, the field theory potential derived from (5.7) is given by (for large N)

$$V_{\text{field theory}} = \frac{1}{g_{\rm YM}^4 N} H_{ijk} \Phi^j \Phi^k \bar{H}^{ilm} \Phi_l^* \Phi_m^* \,. \tag{5.9}$$

By comparing the two expressions we find

$$H_{ijk} = \frac{4\pi\kappa\alpha' \sqrt{g_{\rm YM}^2 N}}{R_0^6} h_{ijk} \,, \tag{5.10}$$

so our formula (5.4) above gives

$$H_{ijk}^{\rm SUGRA}(r) = H_{ijk} + \frac{3R_0^4}{16\pi^2 g_{\rm YM}^2 N(\alpha')^2} H_{ijk}^{(3)} \log\left(\frac{r}{r_0}\right) = H_{ijk} + \frac{3}{16\pi^2} H_{ijk}^{(3)} \log\left(\frac{r}{r_0}\right), \quad (5.11)$$

where $H^{(3)}$ is related to H in the same way (5.2) that $h^{(3)}$ is related to h.

We can compare this with the field theory expression, using the fact that in our normalizations

$$\beta_{H_{ijk}} \equiv \frac{\partial H_{ijk}}{\partial \log(\mu)} = -\frac{1}{2}\left(H_{ijl}\gamma_l^k + H_{ilk}\gamma_j^l + H_{ljk}\gamma_l^i\right), \quad (5.12)$$

and the traceless part of the anomalous dimension matrix is given by (translating (2.4) to the normalizations of (5.7))

$$\gamma_i^j \equiv \frac{\partial \log(Z_i^j)}{\partial \log(\mu)} = -\frac{(N^2-4)}{8N^2\pi^2} H_{ikl}\bar{H}^{jkl}. \quad (5.13)$$

Assuming that the trace of the gamma matrix vanishes this gives for large N

$$H_{ijk}^{field\ theory}(\mu) = H_{ijk} + \frac{3}{16\pi^2} H_{ijk}^{(3)} \log\left(\frac{\mu}{\mu_0}\right) + \cdots, \quad (5.14)$$

where μ is the renormalization scale.

Identifying the radial direction r with the renormalization scale μ via the UV/IR correspondence [30],[21] we find a precise agreement between the weak coupling computation (in field theory) and the strong coupling computation (in supergravity)! This indicates that perhaps the leading terms in the h-perturbation expansion (and in the $1/N$ expansion) in the beta functions, and therefore also in the anomalous dimensions, do not depend on $g_{\rm YM}$ (presumably this is not true for higher order terms in h). Apriori the term in γ proportional to $(h\bar{h})_{\mathbf{8}}$ could have a coefficient with an arbitrary dependence on $\lambda_{\rm YM} = g_{\rm YM}^2 N$, but it seems that it is actually a constant (at least in the large-N limit). The fact that the SUGRA result is independent of $g_{\rm YM}$ seems to follow from the $SL(2,\mathbb{R})$ and $U(1)_Y$ symmetries of the SUGRA action, as discussed in [29]. However, it is surprising that we find precisely the same coefficient in the two limits. This is presumably related to some non-renormalization theorem for the coefficient of this term in γ, but we do not know how to prove this or to relate it to any of the known non-renormalization theorems in $\mathcal{N} = 4$ SYM.

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[21] Putting an IR cutoff on the radial direction at some r_0, and defining the couplings by the values of the fields at r_0, is believed to be equivalent to putting a UV cutoff in the field theory at a scale $\mu_0 \propto r_0$ and defining the couplings at the scale μ_0, though the precise form of this cutoff in the field theory is not known.

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A. Finite groups for $\mathcal{M}_c$

We showed in section 2 that near the $\mathcal{N}=4$ fixed line, the space $\mathcal{M}_c$[22] was parameterized by ten couplings h_{ijk} in the $\mathbf{10}$ representation of SU(3), obeying $(h\bar{h})_{\mathbf{8}}=0$. The constraint has just the right form so that the space of couplings up to global SU(3) transformations is the holomorphic quotient $\mathbf{10}/\,\mathrm{SL}(3,\mathbb{C})$. Here we would like to show that $\mathbf{10}/\,\mathrm{SL}(3,\mathbb{C})=\mathbb{C}^2/\mathcal{T}$, where $\mathcal{T}=\mathrm{SL}(2,\mathbb{Z}_3)$ is embedded in U(2) in a way which we describe later, such that it is a double cover over the tetrahedral group in $\mathrm{SO}(3)\simeq \mathrm{U}(2)/\,\mathrm{U}(1)$. We demonstrate this using some discrete subgroups of the SU(3) gauge group, and then we mention how this result would appear from the usual analysis of gauge invariant coordinates on the moduli space. We will find that $\mathbb{C}^2/\mathcal{T}\simeq\mathbb{C}^2$ as a complex variety, namely, $\mathcal{M}_c$ can be charted by two unconstrained complex coordinates.

The starting point is to define a discrete subgroup $G_1\subset \mathrm{SU}(3)$ such that the adjoint ($\mathbf{8}$) will not have any G_1 invariants, which is equivalent to requiring that G_1 does not commute with any generator, or that its normalizer is discrete. Given such a subgroup and any representation R, its G_1 invariant subspace R_{G_1} is transverse to the gauge orbits, and this is helpful in describing $R/\,\mathrm{SL}(3,\mathbb{C})$. More precisely, R_{G_1} after being divided by a residual discrete gauge group is a subspace of $R/\,\mathrm{SL}(3,\mathbb{C})$, and in our case the inclusion will be shown to be an equality by dimension counting.

We take G_1 to be the "fuzzy torus" or "shift and clock" subgroup defined by two generators U, V obeying $U^3=V^3=1$, $U\,V=\omega\,V\,U$ where $\omega=\exp(2\pi\, i/3)$ is a third root of unity. It is a group of order 27, with a $Z(G_1)\simeq\mathbb{Z}_3$ center generated by ω, and

[22] Throughout this section we will use the shorthand notation $\mathcal{M}_c$ for $\mathcal{M}_c(\mathcal{N}=4)$ and ignore the dimension of this space arising from the gauge coupling constant.

from the defining relations we see that $G_1/Z(G_1) = \mathbb{Z}_3 \times \mathbb{Z}_3$. An explicit representation of G_1 is given by

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ -“shift”},$$
$$V = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}\right] \text{ -“clock”}. \tag{A.1}$$

Actually we can simply ignore the center of SU(3) since both the **10** and the **8** representations, which we will be interested in, have zero triality, and thus are representations of $\mathrm{SU}(3)/\mathbb{Z}_3$. From this point of view $\tilde{G}_1 \equiv G_1/\mathbb{Z}_3 \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathrm{SU}(3)/\mathbb{Z}_3$.

As anticipated, the **8** of SU(3) does not have G_1 invariants.[23] This is equivalent to saying that once we impose G_1 invariance the γ function conditions simplify to the form $\gamma_i^j = \gamma \delta_i^j$.

Looking for G_1 invariants of the **10** it is convenient to represent the **10** by polynomials of degree three. The only polynomials which are invariant under G_1 are[24]

$$O_1 = \sqrt{6}\phi_1\phi_2\phi_3\,,$$
$$O_2 = \frac{1}{\sqrt{3}}(\phi_1^3 + \phi_2^3 + \phi_3^3)\,. \tag{A.2}$$

We arrive at the same operators as in the field theory analysis, eq. (2.5), while the normalizations here were chosen to impose unit length in the natural metric of the symmetric tensor product of ϕ's. Moreover, G_1 is the maximal subgroup that keeps $P = \mathbb{C}^2 = \mathrm{span}(O_1, O_2)$ fixed.[25] We see that the **10** has $2_{\mathbb{C}}$ G_1-invariants, which equals the dimension $10_{\mathbb{C}} - 8_{\mathbb{C}}$ of the quotient $\mathbf{10}/\mathrm{SL}(3,\mathbb{C})$, and so the invariant space covers all of the quotient, up to identifications.

In order to find the identifications on the plane P, which are the main claim of this appendix, we note that fixing a representation of G_1 breaks the SU(3) gauge group to the normalizer $N(G_1) \subset \mathrm{SU}(3)$, (namely $N(G_1)$ is the subgroup that keeps P invariant[26]), and the identifications are given by the action of $N(G_1)/G_1$.

[23] An element $\gamma \in \mathbf{8}$, a traceless 3 by 3 matrix, transforms under G_1 as $\gamma \to g\gamma g^{-1}$. Taking $g = V$ limits γ to be diagonal, and then taking $g = U$ leaves $\gamma = a\delta_i^j$ which vanishes due to the vanishing of the trace.

[24] Invariance under V leaves 4 operators: $\phi_1\phi_2\phi_3$, ϕ_1^3, ϕ_2^3, ϕ_3^3. Then, invariance under U leaves only O_1, O_2 as stated.

[25] For O_1 to be invariant k must be a product of a permutation with a diagonal matrix, while for O_2 to be invariant all three non-zero entries of k must be cubic roots of unity, which amounts to the characterization of G_1.

[26] *Proof:* $N(G_1) = \{n \in \mathrm{SU}(3) | nG_1n^{-1} = G_1\}$. First suppose there is a group element n such that $nP = P$. Then for all $O \in P$ $ngn^{-1}O = O$ so P is invariant under $k = ngn^{-1}$, but G_1 is the maximal subgroup that keeps P invariant and so $k \in G_1$. Conversely, if n is a group element in $N(G_1)$ then for any $g \in G_1$ there is a $g' \in G_1$ such that $gn = ng'$. In particular, for all $O \in P$ $gnO = ng'O = nO$, and so $nO \in P$, namely $nP = P$. QED.

Claim: $N(G_1)/G_1 = \mathrm{SL}(2,\mathbb{Z}_3) = \mathcal{T}$.

Proof. First we will show that $N(G_1)/G_1 \subset \mathrm{SL}(2,\mathbb{Z}_3)$, and then we will write down its generators and verify that it actually equals $\mathrm{SL}(2,\mathbb{Z}_3)$.

Any element $n \in N(G_1)/G_1$ is mapped to $\mathrm{SL}(2,\mathbb{Z}_3)$ by the induced conjugation automorphism of $G_1/\mathbb{Z}_3 = \mathbb{Z}_3 \times \mathbb{Z}_3$, namely an invertible element $M(n)$ of $GL(2,\mathbb{Z}_3)$. This automorphism keeps ω invariant and sends

$$U \to U^{M_{11}}\, V^{M_{12}}\, \omega^{k_1} \tag{A.3}$$

$$V \to U^{M_{21}}\, V^{M_{22}}\, \omega^{k_2} \tag{A.4}$$

where $M(n)_{ij} \in GL(2,\mathbb{Z}_3)$ and $k_1, k_2 \in \mathbb{Z}_3$ are some constants. The commutator $C = UVU^{-1}V^{-1} = \omega \in G_1$ transforms by $C \to \omega^{det(M)}$, thus its invariance actually requires $M \in \mathrm{SL}(2,\mathbb{Z}_3)$. The mapping is actually one to one, namely the only $n \in N(G_1)$ with $M(n) = 1$ are $n \in G_1$.[27]

Now let us construct $N(G_1)$ explicitly by adding to G_1 the generators

$$S = \frac{-i}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \qquad T = \begin{bmatrix} \exp 2\pi i/9 & 0 & 0 \\ 0 & \exp 2\pi i/9 & 0 \\ 0 & 0 & \exp 14\pi i/9 \end{bmatrix}. \tag{A.5}$$

One can check that $M(S)$, $M(T)$ are indeed the usual matrices $S, T \in \mathrm{SL}(2,\mathbb{Z}_3)$ which are known to generate the whole group, thereby completing the proof of the claim.

One checks that P is indeed invariant under S, T, and that on P (that is on the column vector $[O_1\ O_2]$) they are represented by

$$S_O = \frac{i}{\sqrt{3}}\begin{bmatrix} -1 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}, \qquad T_O = \begin{bmatrix} 1 & 0 \\ 0 & \omega^{-1} \end{bmatrix}. \tag{A.6}$$

Let us see why $\mathrm{SL}(2,\mathbb{Z}_3)/\mathbb{Z}_2$ is the tetrahedral group. $\{1,-1\}$ is normal in $\mathrm{SL}(2,\mathbb{Z}_3)$, once we divide by it $\mathrm{SL}(2,\mathbb{Z}_3)/\mathbb{Z}_2$ can be seen to operate on the projective space $\mathbb{Z}_3\mathbf{P}^1 \simeq (\mathbb{Z}_3 - \{0\})/\mathbb{Z}_2$ which has 4 elements, and therefore $\mathrm{SL}(2,\mathbb{Z}_3)/\mathbb{Z}_2 \simeq A_4$, the group of alternating (even) permutations on four elements, which coincides with the tetrahedral group.

The quotient $\mathbf{10}/\,\mathrm{SL}(3,\mathbb{C})$ can be characterized by invariant polynomials, namely, polynomials in $h_{ijk} \in \mathbf{10}$ with symmetric indices and in the antisymmetric tensor ϵ^{ijk}. The lower order invariants can be determined directly — there are no invariants of degree 3 or lower, and there is exactly one of fourth degree,

$$h^4 = h_{i1,i2,i3}\, h_{j1,j2,j3}\, h_{k1,k2,k3}\, h_{l1,l2,l3}\; \epsilon^{i1,j1,k1}\, \epsilon^{i2,j2,l1}\, \epsilon^{i3,k2,l2}\, \epsilon^{j3,k3,l3}\,. \tag{A.7}$$

In order to carry a full analysis of the invariants and proceed to higher invariants, we use the relation $\mathbf{10}/\,\mathrm{SL}(3,\mathbb{C}) = \mathbb{C}^2/\mathcal{T}$ to reduce the problem to that of finding $\mathcal{T}$ invariants on symmetric products of P, namely $\mathrm{Sym}^k(P)$. The number of invariants in a representation is given by the character formula

$$\mathrm{Inv} = \frac{1}{|\mathcal{T}|}\sum_{g\in\mathcal{T}} \chi(g)\,, \tag{A.8}$$

[27] Such an n commutes with U, V and therefore keeps P invariant, and by the maximal property of G_1 we have $n \in G_1$.

where $\chi(g) = \mathrm{tr}(g)$, and the order of the group is $|\mathcal{T}| = 24$. Since the character is constant over conjugacy classes it is useful to determine those to be

$$\begin{array}{|c|ccccccc|}\hline \text{element} & \mathbf{1} & -\mathbf{1} & S,\, S^{-1} & T & T^2 & -T & -T^2 \\ \hline \text{multiplicity} & 1 & 1 & 6 & 4 & 4 & 4 & 4 \\ \text{diagonal form} & (1,1) & (-1,-1) & (i,-i) & (1,\omega) & (1,\omega^2) & (-1,-\omega) & (-1,-\omega^2) \\ \hline \end{array} \tag{A.9}$$

where the diagonal form refers to the diagonalized form of our 2d representation. One finds that the number of invariants of $\mathrm{Sym}^k(P)$ vanishes for odd k, while for even $k = 2j$ it is $\mathrm{Inv}(j) = j/6 + \left[2 + (-)^j 6 + (-)^j 16\, \cos(2\pi j/6)[\cos(2\pi j/3) + \sin(2\pi j/3)/\sqrt{3}]\right]/24$. Once the number of invariants is tabulated one finds that the generating function is given by $\sum \mathrm{Inv}(j)\, q^j = 1/[(1-q^2)(1-q^3)]$, which means that there are two and only two primitive invariants — one at order $k = 2j = 4$ (which we described directly in (A.7)) and one at order $k = 2j = 6$.

Going back to the $\mathrm{SL}(3,\mathbb{C})$ invariants this implies the existence of a second invariant h^6. So there are exactly 2 invariants (coordinates) on this space

$$h^4\,, \qquad h^6\,, \tag{A.10}$$

with no relations among them. Therefore the space is not singular as a complex variety, $\mathbb{C}^2/\mathcal{T} \simeq \mathbb{C}^2$, though it may have a metric singularity. It would be interesting to derive this directly in terms of $\mathrm{SL}(3,\mathbb{C})$ invariants rather than using $\mathcal{T}$.

B. Some SU(3) and SU(4) representations

In this paper we denote the irreducible representation by their dimension, and when ambiguous we use the conventions of [31]. We list here a dictionary of the Dynkin indices (which are directly related to the Young tableaux) for these representations, together with some decompositions. The Dynkin indices are given in parentheses and the commas between them are omitted, as is customary.

SU(3) **representations:**

$$\begin{aligned} \mathbf{3} &= (10) & \mathbf{15'} &= (40) \\ \mathbf{6} &= (20) & \mathbf{21} &= (05) \\ \mathbf{8} &= (11) & \mathbf{24} &= (13) \\ \mathbf{10} &= (30) & \mathbf{27} &= (22) \\ \mathbf{15} &= (21) & \mathbf{28} &= (60) \end{aligned} \tag{B.1}$$

SU(4) **representations and useful decompositions under** $\mathrm{SO}(6) \to \mathrm{SU}(3) \times \mathrm{U}(1)$**:**

$$\begin{aligned} \mathbf{4} &= (100) \to \mathbf{3}_1 + \mathbf{1}_{-3} \\ \bar{\mathbf{4}} &= (001) \to \bar{\mathbf{3}}_{-1} + \mathbf{1}_3 \text{ (by complex conjugation)} \\ \mathbf{6} &= (010) \to \mathbf{3}_{-2} + \bar{\mathbf{3}}_2 \\ \mathbf{10} &= (200) \to \mathbf{6}_2 + \mathbf{3}_{-2} + \mathbf{1}_{-6} \\ \mathbf{15} &= (101) \to \mathbf{8}_0 + \mathbf{3}_4 + \bar{\mathbf{3}}_{-4} + \mathbf{1}_0 \end{aligned}$$

$$
\begin{aligned}
\mathbf{45} &= (210) \;\rightarrow\; \mathbf{15}_4 + \mathbf{10}_0 + \mathbf{8}_0 + \mathbf{6}_{-4} + \mathbf{3}_{-8} + \bar{\mathbf{3}}_{-4} \\
\overline{\mathbf{20}} &= (110) \;\rightarrow\; \mathbf{8}_3 + \mathbf{6}_{-1} + \mathbf{3}_{-5} + \bar{\mathbf{3}}_{-1} \\
\mathbf{20}' &= (020) \;\rightarrow\; \mathbf{6}_{-4} + \bar{\mathbf{6}}_4 + \mathbf{8}_0 \\
\overline{\mathbf{20}''} &= (300) \;\rightarrow\; \mathbf{10}_3 + \mathbf{6}_{-1} + \mathbf{3}_{-5} + \mathbf{1}_{-9} \\
\mathbf{36} &= (201) \;\rightarrow\; \mathbf{15}_1 + \mathbf{8}_{-3} + \mathbf{6}_5 + \mathbf{3}_1 + \bar{\mathbf{3}}_{-7} + \mathbf{1}_{-3} \\
\mathbf{60} &= (120) \;\rightarrow\; \overline{\mathbf{15}}_5 + \mathbf{15}_1 + \mathbf{10}_{-3} + \mathbf{8}_{-3} + \mathbf{6}_{-7} + \bar{\mathbf{6}}_1 \\
\mathbf{64} &= (111) \;\rightarrow\; \mathbf{15}_{-2} + \overline{\mathbf{15}}_2 + \mathbf{8}_6 + \mathbf{8}_{-6} + \mathbf{6}_2 + \bar{\mathbf{6}}_{-2} + \mathbf{3}_{-2} + \bar{\mathbf{3}}_2 \\
\mathbf{84} &= (202) \;\rightarrow\; (\mathbf{27} + \mathbf{8} + \mathbf{1})_0 + \mathbf{15}_4 + \overline{\mathbf{15}}_{-4} + \mathbf{6}_8 + \bar{\mathbf{6}}_{-8} + \mathbf{3}_4 + \bar{\mathbf{3}}_{-4} \\
\mathbf{84}' &= (310) \;\rightarrow\; \mathbf{15}'_1 + \mathbf{24}_5 + \mathbf{15}_1 + \mathbf{10}_{-3} + \mathbf{8}_{-3} + \mathbf{6}_{-7} + \mathbf{3}_1 + \bar{\mathbf{3}}_{-7} \\
\mathbf{105} &= (040) \;\rightarrow\; \mathbf{15}'_{-8} + \overline{\mathbf{24}}_{-4} + \mathbf{27}_0 + \mathbf{24}_4 + \overline{\mathbf{15}'}_8 \\
\mathbf{126} &= (220) \;\rightarrow\; \mathbf{27}_6 + \mathbf{24}_2 + \mathbf{15}'_{-2} + \overline{\mathbf{15}}_2 + \mathbf{15}_{-2} + \mathbf{10}_{-6} + \mathbf{8}_{-6} + \mathbf{6}_{-10} + \bar{\mathbf{6}}_{-2}\,. \qquad \text{(B.2)}
\end{aligned}
$$

To get the U(1) charge normalization of [31], charges must be multiplied by $(-1/3)$.

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